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MATHÉMATIQUE

Sur quelques problèmes d'approximation relatifs à une nouvelle classe d'équations intégro-différentielles

par

D. MANGERON et L. E. KRIVOCHÉINE

Présenté par T. WAŻEWSKI le 17 août 1961

1. Une série de problèmes à la fontière

$$[A(x)\,u'+\lambda B(x)\,u]'+\lambda\,[B(x)\,u'+C(x)\,u]=0 \qquad u|_{\mathrm{Fr}R}=0$$

$$(\mathrm{Fr}R — \text{frontière de } R)$$

dont la nouveauté consiste dans le fait que R est un domaine ,,rectangulaire'' à $2n$ dimensions et le symbole ' dénote la dérivée totale au sens de M. Picone (voir, p.ex., [1]), les problèmes bien posés au sens de J. Hadamard et résolus pour la première fois par l'un des auteurs de cette Note [2]—[4], constitute, parallèlement à d'autres travaux plus récents relatifs aux problèmes à la frontière et aux développements en séries de fonctions caractéristiques correspondantes aux équations non elliptiques, le point de départ des notes élaborées par Yu. M. Bérézanski [5] et consacrées aux problèmes à la frontière pour des opérateurs généraux aux dérivées partielles.

Ia. B. Bykov, S. Vasilache, les auteurs de la présente et d'autres encore ont étudié dans ces dernières années nombre de problèmes, en y comprenant de nombreux problèmes à la frontière relatifs aux équations intégro-différentielles [6]—[9].

Dans [7], par ex., M. S. Vasilache étudie, par la méthode de la traduction dans certaines équations intégrales, les équations intégro-différentielles de la forme

$$\sum_{q=0}^{m}\sum_{p=0}^{n} H_{pq}(x,y)\,\varphi^{(p+q)}_{x^p y^q}(x,y)=f(x,y)+$$
$$+\lambda\int_a^x\int_b^y\sum_{q=0}^{m}\sum_{p=0}^{n} K_{pq}(x,y,u,v)\,\varphi^{(p+q)}_{u^p v^q}(u,v)\,du\,dv,$$

où les fonctions $K(x,y,u,v)$, $H_{pq}(x,y)$, $f(x,y)$ sont continues et dérivables tant qu'il est nécessaire dans le domaine D, défini par les inégalités $a\leqslant u\leqslant x\leqslant a_1$, $b\leqslant v\leqslant y\leqslant b_1$, tandis qu'on suppose vérifiées les conditions aux limites $(\varphi^{5}_{x^{5}}(x,y))_{x=a}$. $(v=0,1,\dots,n-1)$ sont données sur le ségment $b\leqslant y\leqslant b_1$,

$x = a$ et m fois dérivables par rapport à y et $(\varphi^r_{y^r}(x, y))_{y=b} = (\varphi^{n+r}_{x^n y^r}(x, y))_{y=b}$ sont données sur le segment $a \leqslant u \leqslant x \leqslant a_1$, $y = b$ pour $(r = 0, 1, \ldots, m - 1)$.

Nous nous proposons ici de résoudre quelques problèmes d'approximation relatifs aux problèmes suivants:

Problème A

$$(1) \qquad [D^i u]_{x=a} = [D^i u]_{y=b} = 0, \quad (i = 0, 1, \ldots, n - 1),$$

$$(2) \qquad D^n u(x, y) - \lambda (n - 1)!^2 [A(x, y) u(x, y) + B(x, y) D^r u(x, y)] =$$
$$= (n - 1)!^2 [f(x, y) + \lambda \iint_P E(x, y, \xi, \eta) \sum_0^n F_p(\xi, \eta) Du^p(\xi, \eta) \, d\xi, d\eta,$$

où Du est la dérivée totale au sens de M. M. Picone, $Du(x, y) \equiv u' = \dfrac{\partial^2 u(x, y)}{\partial x \, \partial y}$, $A(x, y)$, $B(x, y)$, $E(x, y, \xi, \eta)$, $F_p(\xi, \eta)$ sont des fonctions continues par rapport à leur arguments x, y, ξ, η et pas identiquement nulles dans P_1 et $P - P_1$; $P = \{a \leqslant x, \ \xi \leqslant c; \ b \leqslant y, \ \eta \leqslant d\}$; $P_1 = \{a \leqslant \xi \leqslant x, \ b \leqslant \eta \leqslant y\}$; λ est un paramètre et $(u(x, y)$ en est la solution cherchée.

Problème B

(2) et

$$(3) \qquad u|_L \equiv 0; \ Du|_{L_1} \equiv \ldots \equiv D^{n-1} u|_{L_1} = 0 \left(\begin{matrix} L_1 \equiv \begin{cases} x = a, & b \leqslant y \leqslant d, \\ y = b, & a \leqslant x \leqslant c \end{cases} \\ L \text{ est le contour du domaine } P \end{matrix} \right).$$

2. Si on applique au système (1), (2) l'idée générale de [3] p.p. 544—546, celui-ci se traduit, par la transformation

$$(4) \qquad u(x, y) = \frac{1}{(n-1)!^2} \int_{ab}^{xy}\!\!\int [(x - \xi)(y - \eta)]^{n-1} \varphi(\xi, \eta) \, d\xi \, d\eta \equiv$$
$$\equiv \frac{1}{(n-1)!^2} \iint_{P_1} [(x - \xi)(y - \eta)^{n-1} \varphi(\xi, \eta) \, d\xi \, d\eta,$$

dans l'équation intégrale

$$(5) \qquad \varphi(x, y) - \lambda \iint_P R(x, y, \xi, \eta) \varphi(\xi, \eta) \, d\xi \, d\eta = F(x, y),$$

où

$$R(x, y, \xi, \eta) \equiv \begin{cases} C(x, y, \xi, \eta) + D(x, y, \xi, \eta), & (\xi, \eta) \in P_1 \\ D(x, y, \xi, \eta), & (\xi, \eta) \in P - P_1 \end{cases}$$

et $C(x, y, \xi, \eta)$, $F(x, y)$ et $D(x, y, \xi, \eta)$ sont des fonctions connues tandis que $\varphi(x, y)$ est la fonction qu'on cherche à déterminer.

Supposons qu'on ne connaît pas le noyau résolvant de l'équation intégrale (5). En admettant que la condition

$$(6) \qquad l = 1 - |\lambda_0| \max_P \iint_P |R(x, y, \xi, \eta)| \, d\xi \, d\eta > 0,$$

qui nous assure l'existence et l'unicité de la solution du problème (1), (2), est satisfaite, on peut construire la solution de l'équation (5) sous forme d'une série, à savoir

$$\varphi(x, y) = \sum_{0}^{\infty} \lambda_0^i v_i(x, y), \tag{7}$$

où

$$v_0(x, y) \equiv F(x, y);\; v_i(x, y) \equiv \int\int_P R(x, y, \xi, \eta)\, v_{i-1}(\xi, \eta),\, d\xi\, d\eta, \quad (i = 1, 2, \ldots).$$

Ou peut ensuite prendre, à la suite de la condition (6) qui correspond à la convergence absolue et uniforme de la série (7), pour les solutions exactes et approchées du problème considéré les fonctions

$$u(x, y) = \frac{1}{(n-1)!^2} \int\int_{P_1} [(x-\xi)(y-\eta)]^{n-1} \sum_{0}^{\infty} \lambda_0^i v_i(\xi, \eta)\, d\xi\, d\eta, \tag{8}$$

$$u_s(x, y) = \frac{1}{(n-1)!^2} \int\int_{P_1} [(x-\xi)(y-\eta)]^{n-1} \sum_{0}^{s} \lambda_0^i v_i(\xi, \eta)\, d\xi, d\eta. \tag{9}$$

Soit

$$\beta(x, y, \lambda_0) \equiv \sum_{0}^{s} \lambda_0^i [v_i(x, y) - \lambda_0 \int\int_P R(x, y, \xi, \eta)\, v_i(\xi, \eta)\, d\xi\, d\eta - F(x, y).$$

Alors l'inégalité

$$|u(x, y) - u_s(x, y)| \leqslant \beta\, [(x-a)(y-b)]^n : [l \cdot n!^2],\quad (x, y) \in P, \tag{10}$$

où

$$\beta = \max_P |\beta(x, y, \lambda_0)|,$$

constitute l'évaluation des erreurs correspondantes à la considération des fonctions (9) au lieu de la solution exacte (8) du problème (1), (2).

3. Considérons maintenant le cas où l'on est assuré de l'unicité de la solution du problème (1), (2) mais lorsque la condition (6) n'est pas satisfaite.

Soit un réseau rectangulaire $\Delta x = h_1$, $\Delta y = h_2$, utilisé pour couvrir le domaine P.

Représentons la fonction $\varphi(x, y)$ sous la forme

$$\varphi(x, y) = F(x, y) + \lambda \sum_{i=1}^{k} \sum_{j=1}^{s} d_{ij}\, R(x, y, x_i, y_j)\, \varphi(x_i, y_j) + \lambda \varrho(x, y), \tag{11}$$

où d_{ij} sont des constantes connues, $\varrho(x, y)$ est l'erreur commise par l'admission de ladite formule et $(x_i, y_j) \in P$ est entendu dans le sens $(x_i - 0,\ y_j - 0)$.

En posant $x = x_m$, $y = y_g$ $(m = 1, 2, \ldots, k;\ q = 1, 2, \ldots, s)$, on arrive au système

$$\varphi(x_m, y_q) = F(x_m, y_q) + \lambda \sum_{i=1}^{k} \sum_{j=1}^{s} d_{ij}\, R(x_m, y_q, x_i, y_j)\, \varphi(x_i, y_j) + \lambda \varrho(x_m, y_q). \tag{12}$$

Dans le cas où

$$\det \|d_{ij} R(x_m, y_q, x_i, y_j)\| \neq 0, \quad (i = 1, 2, \ldots, k; \quad j = 1, 2, \ldots, s)$$

on trouve par la voie ordinaire

$$(13) \quad \varphi(x_i, y_j) = \alpha_{ij}(\lambda) + \lambda \sum_{m=1}^{k} \sum_{q=1}^{s} \delta_{ijmq}(\lambda)\, \varrho(x_m, y_q), \; (i=1,2,\ldots,k;\; j=1,2,\ldots,s).$$

Par conséquent, on peut considérer la fonction

$$(14) \quad \bar{u}(x, y) = [1:(n-1)!^2] \iint_{P_1} [(x-\xi)(y-\eta)]^{n-1} \varphi(\xi, \eta)\, d\xi\, d\eta \equiv$$

$$\equiv [1:(n-1)!^2] \sum_{i=1}^{k} \sum_{j=1}^{s} d_{ij}\, \alpha_{ij}(\lambda) \iint_{P_1} [(x-\xi)(y-\eta)]^{n-1} R(\xi, \eta, x_i, y_j\, d\xi\, d\eta)$$

comme une solution approchée du problème (1), (2).

On obtient, en utilisant (5),

$$(15) \quad \varrho(x, y) = \iint_{P} R(x, y, \xi, \eta)\, \varphi(\xi, \eta)\, d\xi\, d\eta -$$

$$- \sum_{i=1}^{k} \sum_{j=1}^{s} d_{ij} R(x, y, x_i, y_j)\, \varphi(x_i, y_j) \equiv \gamma(x, y) + \lambda \iint_{P} R_1(x, y, \xi, \eta)\, \varphi(\xi, \eta)\, d\xi\, d\eta.$$

On a en outre

$$(16) \quad \begin{cases} |\varphi(x, y) - \bar{\varphi}(x, y)| \leqslant |\lambda| \max\limits_{P} |\varrho(x, y)| \cdot \left[1 + \sum\limits_{i=1}^{k} \sum\limits_{j=1}^{s} d_{ij}\right]. \\ \sum\limits_{m=1}^{k} \sum\limits_{q=1}^{s} |\delta_{ijmq}(\lambda)\, R(x_m, y_q, x_i, y_j)| \equiv \max |\varrho(x, y)| \cdot w_1(\lambda). \end{cases}$$

Alors l'inégalité

$$(17) \quad \begin{cases} |u(x, y) - \bar{u}(x, y)| \leqslant w_1(\lambda)\, w_2(\lambda)\, [(x-a)(y-b)]^n : n!^2, \\ \qquad (x, y) \in P; \quad 0 \leqslant |\lambda| \leqslant h, \end{cases}$$

constitute l'évaluation cherchée de l'erreur, où $w_2(\lambda)$ a été trouvée, grâce à l'utilisation des relations (15), (16) sous la forme

$$(18) \quad \max_{P} |\varrho(x, y)| \leqslant w_2(\lambda), \quad 0 = |\lambda| \leqslant h.$$

4. La méthode d'évaluation de l'erreur qu'on rencontre dans les problèmes d'approximation considérés, utilisée dans le par. 3 peut être appliquée sans modifications essentielles lorsqu'on considère le problème à la frontière pour les

équations intégro-differentielles aux dérivées totales au sens de M. M. Picone (3), (2) $r = p = 0$, qu' on traduit au préalable dans l'équation équivalente

$$
\begin{aligned}
(19)\qquad u(x, y) = N(x, y, \xi, \eta)\, \{ f(\xi, \eta) + \lambda\, [A(\xi, \eta) + B(\xi, \eta)]\, u(\xi, \eta) + \\
+ \lambda \iint\limits_{P} E(\xi, \eta, t, \tau)\, F_0(t, \tau)\, u(t, \tau)\, dt\, d\tau \}\, d\xi\, d\eta \equiv \\
\equiv \Phi(x, y) + \lambda \iint\limits_{P} W(x, y, \xi, \eta)\, u(\xi, \eta)\, d\xi\, d\eta,
\end{aligned}
$$

où nous nous sommes servi de la fonction $M(x, y, \xi, \eta) \equiv N(x, y, \xi, \eta) : (n-1)!^2$, construite par l'un de nous dans [10], à savoir

$$
(M)\ M(x, y, \xi, \eta) \equiv \begin{cases}
\dfrac{[(x-a)(c-\xi)(y-b)(d-\eta)]^{n-1}}{[(c-a)\ (d-b)]^{n-1}\,(n-1)!^2} & x \leqslant \xi,\ y \leqslant \eta \\[2ex]
\left[\dfrac{[(x-a)\ (c-\xi)^{n-1}]}{(c-a)^{n-1} -} - (x-\xi)^{n-1}\right] \dfrac{[(y-b)(d-\eta)]^{n-1}}{(d-b)^{n-1}\,(n-1)!^2} & \\
 & x \leqslant \xi,\ y \leqslant \eta \\[2ex]
\dfrac{[(x-a)(c-\xi)]^{n-1}}{(c-a)^{n-1}\,(n-1)!^2} \left[\dfrac{[(y-b)\ (d-\eta)]^{n-1}}{(d-b)^{n-1}} - (y-\eta)^{n-1}\right] \equiv & \\
 & x \leqslant \xi,\ y \geqslant \eta \\[2ex]
\left[\dfrac{[(x-a)\ (c-\xi)]^{n-1}}{(c-a)^{n-1}} - (x-\xi)^{n-1}\right] \times & \\
\quad \times \left[\dfrac{[(y-b)\ (d-\eta)]^{n-1}}{(d-b)^{n-1}} - (y-\eta)^{n-1}\right] & x \geqslant \xi,\ y \geqslant \eta \\[2ex]
\equiv N(x, y, \xi, \eta) : (n-1)!^2 &
\end{cases}
$$

Un article contenant divers détails et extensions concernant ce sujet sera publié dans le Bulletin de l'Institut Polytechnique de Jassy.

Note de la Redaction. La communication ci-dessus a été dédicacée par les auteurs à M. le Professeur T. Kotarbiński, President de l'Académie Polonaise des Sciences pour honorer son 75-ième anniveısaire.

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OUVRAGES CITÉS

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MATHÉMATIQUE

Propriétés des intégrales itérées singulières dans l'espace

par

W. POGORZELSKI

Présenté par A. ZYGMUND le 11 juillet 1961

Cette note concerne une transformation de l'intégrale itérée des deux fonctions à singularités fortes. Cette intégrale a été étudiée par Giraud [1] sous l'autre hypothèse. Nous employons dans le travail présent les mêmes symboles que dans nos travaux [2] et [3].

THÉORÈME 1. *Si $G(x, u)$ et $L(x, u)$ sont deux fonctions définies en tout point $x \neq (0, \ldots, 0)$ de l'espace euclidien à n dimensions et pour u dans le domaine Ω_0 de cet espace par les formules*

$$(1)\qquad G(x, u) = \frac{g(x', u)}{|x|^n}; \qquad L(x, u) = \frac{l(x', u)}{|x|^n}$$

(x' étant une projection du point x sur la surface ω d'une sphère centrée en $(0, \ldots, 0)$ de rayon unité, donc $x = |x|\, x'$) où les fonctions $g(x', u)$ et $l(x', u)$, définies dans la région $[x' \in \omega,\ u \in \Omega_0]$, vérifient les conditions

$$(2)\qquad \int\limits_{\omega} g(x', u)\, dx' = 0; \qquad \int\limits_{\omega} l(x', u)\, dx' = 0,$$

$$(3)\qquad \begin{cases} |g(x', u) - g(\tilde{x}', \tilde{u})| < \text{const}\, [|x' - \tilde{x}'|^{h_\omega} + |u - \tilde{u}|^{h_1}] \\ |l(x', u) - l(\tilde{x}', \tilde{u})| < \text{const}\, [|x' - \tilde{x}'|^{h_\omega} + |u - \tilde{u}|^{h_1}] \end{cases}$$

quels que soient les points u et $\tilde{u}$ dans Ω_0; si, en outre, $f(x)$ est une fonction définie dans la région Ω, de classe $\mathfrak{H}_a^h$ $(h < (h_\omega, h_1))$ relativement aux surfaces de discontinuité $S_0, S_1, \ldots, S_p$ (voir [1]), *alors l'égalité des intégrales itérées suivantes est vraie*

$$(4)\qquad \int\limits_{\Omega} \Psi(x, z) \Big[\int\limits_{\Omega} G(z - y, z)\, dy \Big] dz = \int\limits_{\Omega} \Big[\int\limits_{\Omega} \Psi(x, z)\, G(z - y, z)\, dz \Big] dy,$$

où l'on a posé

$$(5)\qquad \Psi(x, z) = L(x - z, x)\, [f(z) - f(x)];$$

les intégrales par rapport à y (ou z) contiennent une forte singularité $z = y$ et ont le sens de Cauchy; x est un point fixé à l'intérieur de Ω; la fonction $\Psi(x, z)$ admet une singularité faible pour $z = x$ et les singularités faibles aux surfaces S_ν.

Démonstration. La fonction $\varphi(z)$, définie en tout point z du domaine Ω_0 par l'intégrale singulière

$$(6) \qquad \varphi(z) = \int_\Omega G(z-y, z)\, dy = \lim_{\eta\to 0} \int_{\Omega - \Pi(z,\eta)} G(z-y, z)\, dy,$$

appartient à la classe $\mathfrak{H}_\sigma^h$, σ étant un nombre positif arbitrairement petit. On a désigné par $\Pi(z, \eta)$ une sphère de rayon η, centrée en z. Le membre gauche de l'égalité (4) a donc un sens déterminé en tout point $x \in \Omega$. Nous allons démontrer la propriété limite suivante

$$(7) \quad \lim_{\eta\to 0} \left\{ \int_\Omega \Psi(x, z) \Big[\int_{\Omega - \Pi'(z,\eta)} G(z-y, z)\, dy \Big] dz \right\} = \int_\Omega \Psi(x, z) \Big[\int_\Omega G(z-y, z)\, dy \Big] dz$$

où $\Pi'(z, \eta) = \Pi(z, \eta) \times \Omega$ est un ensemble de tous les points communs au domaine Ω et à la sphère $\Pi(z, \eta)$. Dans ce but il suffit de démontrer que l'intégrale itérée singulière

$$(8) \qquad J(x, \eta) = \int_\Omega \Psi(x, z) \Big[\int_{\Pi'(z,\eta)} G(z-y, z)\, dy \Big] dz$$

tend vers zéro si $\eta \to 0$. Dans le cas $\eta \leqslant |z - z_s|$ on a $\Pi' = \Pi$ et par conséquent, d'après (2),

$$(9) \qquad \int_{\Pi'(z,\eta)} G(z-y, z)\, dy = 0\,.$$

Dans le cas $\eta > |z - z_s|$, on aura

$$(10) \qquad \int_{\Pi'(z,\eta)} G(z-y, z)\, dy = \int_{\Pi'(z,\eta) - \Pi_0(z)} G(z-y, z)\, dy$$

$\Pi_0(z)$ désignant une sphère, centrée en z, tangente au point z_0 sur S_0, où la distance $|z - z_0|$ atteint sa borne inférieure. Nous aurons donc une limitation

$$(11) \qquad \Big| \int_{\Pi'(z,\eta)} G(z-y, z)\, dy \Big| < \frac{\text{const}}{|z - z_0|^\sigma}$$

indépendante de η, où σ est un nombre positif arbitrairement petit. Remarquons maintenant qu'en vertu des formules (1), (5) et la supposition $f(x) \in \mathfrak{H}_\alpha^h$, la fonction $\Psi(x, z)$ vérifie une inégalité

$$(12) \qquad |\Psi(x, z)| < \frac{\text{const}}{|x - z|^{n-h}\, |z - z_s|^{\alpha+h}}$$

x étant fixé dans Ω.

D'après (11) et (12), à tout nombre positif ε on peut faire correspondre une couche $C_\varepsilon^{(0)}$ située entre la surface S_0 et la surface S_ε à l'intérieur de Ω dont les points sont situés si près de la surface S_0 qu'on a

$$(13) \qquad \Big| \int_{C_\varepsilon^{(0)}} \Psi(x,z) \Big[\int_{\Pi'(z,\eta)} G(z-y,z)\,dy \Big] dz \Big| < \varepsilon .$$

La couche $C_\varepsilon^{(0)}$ étant fixée, la borne inférieure de la distance $|z-z_0|$ pour $z \in \Omega - C_\varepsilon^{(0)}$, admet une valeur positive δ_ε et nous aurons l'égalité (9) si $z \in \Omega - C_\varepsilon^{(0)}$ et $\eta < \delta_\varepsilon$. En somme on aura

$$|J(x,\eta)| < \varepsilon, \quad \text{si} \quad \eta < \delta_\varepsilon,$$

par conséquent la propriété (7) est prouvée.

Remarquons maintenant que l'intégrale itérée (7) entre les accolades, étendue à la région $[y \in \Omega - \Pi'(z,\eta);\ z \in \Omega]$ ne contient qu'une singularité faible $z = x$ et les singularités faibles aux points des surfaces S_ν. Nous pouvons donc écrire l'égalité suivante

$$(14) \quad \int_\Omega \Psi(x,z) \Big[\int_{\Omega - \Pi'(z,\eta)} G(z-y,z)\,dy \Big] dz = \int_\Omega \Big[\int_{\Omega - \Pi'(y,\eta)} \Psi(x,z)\, G(z-y,z)\,dz \Big] dy$$

où $\Pi'(y,\eta) = \Pi(y,\eta) \times \Omega$, $\Pi(y,\eta)$ désignant une sphère de rayon η, centrée en y. Démontrons la propriété limite

$$(15) \quad \lim_{\eta \to 0} \Big\{ \int_\Omega \Big[\int_{\Omega - \Pi'(z,\eta)} \Psi(x,z)\, G(z-y,z)\,dz \Big] dy \Big\} = \int_\Omega \Big[\int_\Omega \Psi(x,z)\, G(z-y,z)\,dz \Big] dy,$$

où l'intégrale relative à z admet une singularité forte pour $z = y$. Pour démontrer d'abord son existence, écrivons la décomposition évidente

$$(16) \quad \int_{\Omega - \Pi(y,\zeta)} \Psi(x,z)\, G(z-y,z)\,dz = \int_{\Pi(y,\eta) - \Pi(y,\zeta)} [\Psi(x,z) - \Psi(x,y)]\, G(z-y,z)\,dz +$$
$$+ \int_{\Omega - \Pi(y,\eta)} \Psi(x,z)\, G(z-y,z)\,dz + \int_{\Pi(y,\eta) - \Pi(y,\zeta)} \Psi(x,y)\, [G(z-y,z) - G(z-y,y)]\,dz$$

en admettant que

$$(16') \qquad 0 < \zeta < \eta \leqslant \min\Big(|y - y_s|,\ \frac{1}{2}|x-y| \Big).$$

Nous avons ensuite

$$(17) \quad \Psi(x,z) - \Psi(x,y) = [L(x-z,x) - L(x-y,x)]\,[f(z) - f(x)] +$$
$$+ L(x-y,x)\,[f(z) - f(y)]$$

d'où, en profitant de l'inégalité (voir [1])

$$(18) \qquad |L(x-z,x) - L(x-y,x)| < \text{const} \Big[\frac{|y-z|}{|x-y|^{n+1}} + \frac{|y-z|^{h_\omega}}{|x-y|^{n+h_\omega}} \Big]$$

si $z \in \Pi(y, \eta)$ et $|x-y| \geqslant 2\eta$, on aura l'inégalité

$$(19) \qquad |\Psi(x, z) - \Psi(x, y)| < \frac{\text{const}\, |y-z|^h}{|x-y|^n\, |z-z_s|^{\alpha+h}}$$

si $|y-y_s| \geqslant |z-z_s|$. Dans le cas $|y-y_s| < |z-z_s|$ il faut remplacer au dénominateur (19) la distance $|z-z_s|$ par $|y-y_s|$, le point x étant fixé à l'intérieur de Ω. En faisant tendre le rayon ζ vers zéro dans la formule (16), on constate, d'après (12) et (19), l'existence des intégrales singulières suivantes et la vérité de la décomposition

$$(20) \qquad F(x, y) = \int_{\Omega} \Psi(x, z)\, G(z-y, z)\, dz =$$

$$= \int_{\Pi(y,\eta)} [\Psi(x, z) - \Psi(x, y)]\, G(z-y, z)\, dz + \int_{\Omega-\Pi(y,\eta)} \Psi(x, z)\, G(z-y, z)\, dz +$$

$$+ \int_{\Pi(y,\eta)} \Psi(x, y)\, [G(z-y, z) - G(z-y, y)]\, dz$$

si $0 < \eta \leqslant \min(|y-y_s, \frac{1}{2}|x-y|)$.

Le point x étant fixé à l'intérieur de Ω, étudions l'illimitation de la fonction (20), si $|y-y_s| \to 0$, et $|x-y| \to 0$. Considérons donc un système $\mathcal{E}$ de couches situées dans le domaine $\Omega_{(x)}$ contenant le point fixé x, suffisamment proche des surfaces S_ν limitant ce domaine $\Omega_{(x)}$ pour que le point x soit extérieur à $\mathcal{E}$ et pour qu'on ait $|y-y_s| < \frac{1}{2}|x-y|$ si $y \in \mathcal{E}$.

En posant $\eta = |y-y_s|$ dans la formule (20), nous aurons alors en tout point $y \in \mathcal{E} + (\Omega - \Omega_{(x)})$ la décomposition

$$(21) \qquad F(x, y) = \int_{\Pi_s(y)} [\Psi(x, z) - \Psi(x, y)]\, G(z-y, z)\, dz +$$

$$+ \int_{\Omega-\Pi_s(y)} \Psi(x, z)\, G(z-y, z)\, dz + \int_{\Pi_s(y)} \Psi(x, y)\, [G(z-y, z) - G(z-y, y)]\, dz$$

où $\Pi_s(y)$ désigne la sphère, centrée en y, de rayon $|y-y_s|$. Soit encore une sphère $\Delta(x, \varrho_0)$, centrée en x, de rayon ϱ_0, fixé suffisamment petit pour que cette sphère n'aie pas de points communs avec la sphère variable $\Pi_s(y)$, quel que soit y dans $\mathcal{E}$. En raisonnant d'une façon analogique que dans nos travaux [2] et [4], nous déduirons, par la méthode de transformation homothétique, en vertu des formules (12), (19), (21), une limitation à faible singularité

$$(22) \qquad |F(x, y)| < \frac{\text{const}}{|x-y|^n} \int_{\Pi_s(y)} \frac{dz}{|z-z_\Pi|^{\alpha+h}\, |y-z|^{n-h}} +$$

$$+ \frac{\text{const}}{|x-y|^{n-h}\, |y-y_s|^{\alpha+h}} \int_{\Pi_s(y)} \frac{dz}{|z-y|^{n-h}} + \frac{\text{const}}{\varrho_0^{n-h}} \int_{\Omega-\Pi_s(y)-\Delta(x,\varrho_0)} \frac{dz}{|y-z|^n\, |z-z_s|^{\alpha+h}} +$$

$$+ \frac{\text{const}}{\delta_{zy}^{n+\alpha+h}} \int_{\Delta(x,\varrho_0)} \frac{dz}{|x-z|^{n-h}} < \frac{\text{const}}{|y-y_s|^{\alpha+h}\, |x-y|^{n-h}}$$

si $y \in \mathcal{E} + \Omega - \Omega_{(x)}$; on a désigné $\delta_{zy} = \inf |z-y|$ si $z \in \Delta(x, \varrho_0)$ et $y \in \mathcal{E} + \Omega - \Omega_{(x)}$.

Dans le cas $y \in \Omega_{(x)} - \mathcal{E}$ nous pouvons fixer une constante positive c, inférieure à l'unité, pour qu'on ait

$$c\,|x-y| < \tfrac{1}{2}\,|y-y_s| \quad \text{et nous posons dans (20)} \quad \eta = c\,|x-y|.$$

Nous aurons alors

$$(23)\qquad |F(x,y)| < \frac{\text{const}}{|x-y|^n}\int_0^\eta \varrho^{h-1}\,d\varrho + \frac{\text{const}}{|y-y_s|^{\alpha+h}\,|x-y|^{n-h}}\int_0^\eta \varrho^{h-1}\,d\varrho +$$

$$+\Bigg[\int\limits_{[\Omega(x)-\mathcal{E}-\Pi]} + \int\limits_{[\mathcal{E}+\Omega-\Omega(x)]}\Bigg]\frac{\text{const}\,dz}{|x-z|^{n-h}\,|z-y|^n\,|z-z_s|^{\alpha+h}} < \frac{\text{const}}{|x-y|^{n-h}\,|y-y_s|^{\alpha+h}}\,.$$

En somme, d'après (22) et (23), la fonction (20) vérifie une limitation aux singularités faibles

$$(24)\qquad \left|\int\limits_\Omega \Psi(x,z)\,G(z-y)\,dz\right| < \frac{\text{const}}{|x-y^{n-h}\,|y-y_s|^{\alpha+h}}$$

$(x \neq y)$. Il en résulte l'existence de l'intégrale itérée figurant à droite dans l'égalité (15). Pour démontrer la propriété (15), il suffit de démontrer que l'intégrale

$$(25)\qquad I(x) = \int\limits_\Omega\Bigg[\int\limits_{\Pi'(y,\eta)} \Psi(x,z)\,G(z-y)\,dz\Bigg]\,dy$$

tend vers zéro si $\eta \to 0$. Ce fait résulte des limitations que nous avons rencontré dans les raisonnements précédents qui ont conduit à l'inégalité (24).

Théorème 2. *Sous les mêmes hypothèses que dans le Théorème* 1, *on a l'inégalité des intégrales itérées singulières suivantes*

$$(26)\qquad \int\limits_\Omega G(x-z,x)\Bigg[\int\limits_\Omega \Psi(z,y)\,dy\Bigg]\,dz = \int\limits_\Omega\Bigg[\int\limits_\Omega G(x-z,x)\,\Psi(z,y)\,dz\Bigg]\,dy.$$

La démonstration de ce théorème est analogique à la précédente.

Théorème 3. *Sous les hypothèses précédentes, concernant les fonctions* L, G, f, *la transformation suivante des intégrales singulières itérées est vraie*,

$$(27)\qquad \int\limits_\Omega L(x-z,x)\Bigg[\int\limits_\Omega G(z-y,z)\,f(y)\,dy\Bigg]\,dz = \Phi(x)f(x) +$$

$$+\int\limits_\Omega\Bigg[\int\limits_\Omega L(x-z,x)\,G(z-y,z)\,dz\Bigg]f(y)\,dy$$

x étant fixé à l'intérieur de Ω et en admettant que la dernière intégrale singulière a un sens; la fonction $\Phi(x)$, déterminée en tout point $x \in \Omega$, ne dépend pas de la fonction $f(x)$.

Démonstration. Remarquons d'abord que les intégrales, figurant au membre gauche de (27), ont un sens déterminé en tout point $x \in \Omega$. En effet, en vertu des nos travaux [2] et [3], si $f(y) \in \mathfrak{H}_a^h$ $(h < h_\omega)$, on a

$$\int_\Omega G(z-y, z) f(y)\, dy \in \mathfrak{H}_a^h,$$

donc le membre gauche (27) a un sens déterminé en tout point intérieur $x \in \Omega$ et appartient aussi à la classe $\mathfrak{H}_a^h$. Pour étudier l'existence de l'intégrale du membre droit de (27), nous constatons, d'après (18), par analogie avec (20), l'existence de l'intégrale singulière suivante et la vérité de la décomposition $(x \neq y)$

$$(28)\qquad H(x, y) = \int_\Omega L(x-z, x)\, G(z-y, z)\, dz =$$

$$= \int_{\Pi(y,\eta)} [L(x-z, x) - L(x-y, x)]\, G(z-y, z)\, dz + \int_{\Omega-\Pi(y,\eta)-\Pi(x,\zeta)} L(x-z, x)\, G(z-y, z)\, dz +$$

$$+ \int_{\Pi(x,\zeta)} L(x-z, x)\, [G(z-y, z) - G(x-y, x)]\, dz +$$

$$+ \int_{\Pi(y,\eta)} L(x-y, x)\, [G(z-y, z) - G(z-y, y)]\, dz$$

où les rayons arbitraires η, ζ vérifient les inégalités

$$0 < \eta \leqslant \min(|y-y_s|, \quad \tfrac{1}{2}|x-y|);$$

$$0 < \zeta \leqslant \min(|x-x_s|, \quad \tfrac{1}{2}|x-y|).$$

D'une façon analogique que pour l'intégrale (20) on en tire une limitation

$$(29)\qquad |H(x, y)| < \frac{\text{const}}{|x-y|^n}; \qquad (n \geqslant 2)$$

qui malheureusement contient de nouveau une singularité forte. L'itération des intégrales singulières dans l'espace n'abaisse pas le degré de singularité, autrement que pour l'intégrale linéaire dans la transformation connue de Poincaré — Bertrand. Cette circonstance a déjà été signalée par Giraud [1]. Or, si nous admettons que le noyau singulier $H(x, y)$. défini par la formule (28), est tel que l'intégrale singulière

$$\int_\Omega H(x, y) f(y)\, dy$$

existe au sens de Cauchy, alors la transformations (27) est vraie. En effet, en s'appuyant sur les Théorèmes 1 et 2, nous pouvons écrire une suite des égalités suivantes

$$(30)\qquad \int_\Omega L(x-z, x) \Big[\int_\Omega G(z-y, z) f(y)\, dy\Big] dz =$$

$$= \int_\Omega L(x-z, x) \Big[\int_\Omega G(z-y, z)\, [f(y) - f(z)]\, dy\Big] dz +$$

$$+ \int_\Omega L(x-z,x)\,[f(z)-f(x)]\left[\int_\Omega G(z-y,z)\,dy\right]dz + f(x)\int_\Omega L(x-z,x)\times$$

$$\times\left[\int_\Omega G(z-y,z)\,dy\right]dz = \int_\Omega\left\{\int_\Omega L(x-z,x)\,G(z-y,z)\,[f(y)-f(z)]\,dz\right\}dy +$$

$$+ \int_\Omega\left\{\int_\Omega L(x-z,x)\,[f(z)-f(x)]\,G(z-y,z)\,dz\right\}dy + f(x)\int_\Omega L(x-z,x)\times$$

$$\times\left[\int_\Omega G(z-y,z)\,dy\right]dz = \int_\Omega\left\{\int_\Omega L(x-z,x)\,G(z-y,z)\,dz\right\}f(y)\,dy + \Phi(x)\,f(x)$$

en posant

$$\Phi(x) = \int_\Omega L(x-z,x)\left[\int_\Omega G(z-y,z)\,dy\right]dz - \int_\Omega\left[\int_\Omega L(x-z,x)\,G(z-y,z)\,dz\right]dy. \tag{31}$$

D'après nos hypothèses, cette fonction a une valeur déterminée en tout point $x \in \Omega_0$ et ne dépend pas de la fonction $f(x)$.

INSTITUT MATHÉMATIQUE, ACADÉMIE POLONAISE DES SCIENCES
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OUVRAGES CITÉS

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MATHEMATICS

On Analytic Spaces

by

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1. Notation and terminology

For convenience all spaces under consideration are supposed to be completely regular.

A mapping f of a space X onto a space Y will be called perfect if f is both continuous and closed (the images of closed sets are closed) and the inverse images of points are compact.

If $\mathcal{D}$ is a class of spaces, then $A\mathcal{D}$ will be used to denote the class of all continuous images of spaces from $\mathcal{D}$. If $A\mathcal{D} \supset \mathcal{C}$, then $\mathcal{D}$ will be called a A-base of $\mathcal{C}$. If $A\mathcal{D} = \mathcal{D}$, then $\mathcal{D}$ will be called A-closed or closed under A. The symbol $A^{-1}\,\mathcal{D}$ will be used to denote the class of the inverse images under continuous mappings of spaces from $\mathcal{D}$. The meaning of terms A^{-1}-base and A^{-1}-closed is clear. Using perfect mappings instead of continuous mappings we obtain the definitions of a P-base, a P^{-1} base, P-closed and P^{-1}-closed.

A class $\mathcal{D}$ of spaces will be called countably productive if the topological product of every countable subclass of $\mathcal{D}$ belongs to $\mathcal{D}$. Finally, a class $\mathcal{D}$ will be called F-hereditary if closed subspaces of spaces from $\mathcal{D}$ belong to $\mathcal{D}$.

A centred family is a family with the finite intersection property. If $\mathfrak{M}$ is a family of subsets of a space X, then $\overline{\mathfrak{M}}^X$, or merely $\overline{\mathfrak{M}}$, will be used to denote the family consisting from closures of all sets from $\mathfrak{M}$.

2. Examples

a) The class $\mathcal{K}$ of all compact spaces is A-closed, hence P-closed, P^{-1}-closed, productive and F-hereditary. By Čech—Stone mapping theorem, the class of Čech—Stone compactifications of discrete spaces is a A-base of $\mathcal{K}$.

b) From a) it follows at once that the class $\mathcal{K}_\sigma$ of all σ-compact spaces is A-closed, P-closed, P^{-1}-closed and F-hereditary, but not countably productive. A space X belongs to $\mathcal{K}_\sigma$ if and only if X is a F_σ in every space $Y \supset X$. The class of locally compact spaces from $\mathcal{K}_\sigma$ is a A-base of $\mathcal{K}_\sigma$ but not a P-base of $\mathcal{K}_\sigma$ (see the following example).

c) The class of locally compact spaces is P-closed, P^{-1}-closed and F-hereditary but not A-closed, nor A^{-1}-closed nor countably productive. Moreover, every space is the image under a continuous mapping of a discrete space.

3. E-spaces

A space X will be called a E-space, if X is a $F_{\sigma\delta}$ in some of its compactifications, or, equivalently, if X is a $K_{\sigma\delta}$ in some $Y \supset X$. From the Čech—Stone mapping theorem it follows at once that a space X is a E-space if and only if X is a $F_{\sigma\delta}$ in $\beta(X)$. The class of all E-spaces will be denoted by $\mathcal{E}$.

THEOREM 1. *A necessary and sufficient condition for a space X to be a E-space is that there exists a complete countable collection* (see [4]) *of closed countable coverings of X.*

If $\mathfrak{a}_n$, $n = 1, 2, \ldots$, are countable families consisting of compact subspaces of a space $Z \supset X$ with $X = \bigcap_{n=1}^{\infty} \bigcup \mathfrak{a}_n$, then $\{\mathfrak{a}_n \cap X\}$ is a complete countable collection of closed countable coverings of X. Thus, the condition is necessary. To prove the sufficiency, one must first show that every space satisfying the condition is a Lindelöf space, and, consequently, a normal one, and then one can prove

$$\bigcap_{n=1}^{\infty} \bar{\mathfrak{a}}_n^{\beta(X)} = X, \tag{1}$$

where $\{\mathfrak{a}_n\}$ is a complete sequence of closed countable coverings of X. The proof of (1) is based upon the following property of the Čech—Stone compactification of a normal space X: If $x \in \beta(X) - X$, then the family of all closed subsets of X the closure of which contain x is centred.

Using Theorem 1 one can prove the following result.

THEOREM 2. *The class $\mathcal{E}$ is P^{-1}-closed, F-hereditary and countably productive. Every E-space is a Lindelöf space and hence a normal one.*

THEOREM 3. *A space X belongs to $\mathcal{E}$ if and only if X is homeomorphical with a closed subset of the topological product of a countable number of σ-compact spaces.*

Proof. Since, clearly, every σ-compact space belongs to $\mathcal{E}$, the sufficiency follows from Theorem 2. Conversely, if

$$X = \bigcap_{n=1}^{\infty} F_n, \tag{2}$$

where F_n are F_σ-subsets of $\beta(X)$, then

$$f(x) = \{x, x, \ldots\}$$

is a homeomorphical mapping of X into the topological product F of σ-compact spaces F_n. From (2) it follows at once that $f[X]$ is a closed subset of F.

THEOREM 4. *The class of all topologically complete* (*in the sense of E. Čech*) *spaces from $\mathcal{E}$ is a A-base of $\mathcal{E}$.*

Proof. Let $X \in \mathcal{E}$ and let $\{\mathfrak{F}_n\}$ be a complete sequence of closed countable coverings of X. Let us define a new topology for the set X such that the family of

all open sets in X and all the sets from $\mathfrak{F} = \bigcup_{n=1}^{\infty} \mathfrak{F}_n$ is an open sub-base. Denote this space by Y and consider the space

(3) $$Z = \bigcap_{n=1}^{\infty} \bigcup \overline{\mathfrak{F}}_n^{\beta(Y)}.$$

It is easy to see that Z is a topologically complete space from E and the restriction of the Čech—Stone mapping of $\beta(Y)$ onto $\beta(X)$ to Z is a mapping onto X.

Note. The class of all topologically complete spaces (in the sense of E. Čech) from $\mathcal{E}$ is coincident with the class of all $N(\aleph_0)$ — spaces (see [4]).

4. Analytic spaces

The following modification of Choquet's definition (see [2]) will be used.

Definition 1. The class $A\mathcal{E}$ will be denoted by $\mathcal{A}$ and its elements will be called analytic spaces.

From Definition 1 and the theorems of Section 3 we obtain at once the following result.

Theorem 5. *The class $\mathcal{A}$ is A-closed, F-hereditary and countably productive. The class of all topologically complete E-spaces is a A-base of $\mathcal{A}$. Every analytic space is a Lindelöf space and, hence, a normal one.*

Let us denote by Σ the topological product N^N, where N is the discrete space of all positive integers. The elements of Σ are infinite sequences of positive integers and the topology of Σ is the topology of pointwise convergence. Clearly, Σ is a separable metrizable topologically complete space from $\mathcal{E}$. Let S be the set of all finite sequences of positive integers. If $s \in S$, $s = \{s_1, \ldots, s_n\}$, and $\sigma \in \Sigma$, $\sigma = \{\sigma_1, \sigma_2, \ldots\}$, then the symbol $s \prec \sigma$ will be used to express that s is a section of σ, that means $s_i = \sigma_i$ for all $i \leqq n$. The number n will be called the length of s and the set of all $s \in S$ of lenght n will be denoted by S_n.

A determining system in a family of sets $\mathfrak{M}$ is a mapping $M = \{M(s)\}$ of S to $\mathfrak{M}$. The nucleus of a determining system M is the set

(4) $$\mathfrak{a}(M) = \bigcup_{\sigma \in \Sigma} \bigcap_{s \prec \sigma} M(s).$$

The family of all $\mathfrak{a}(M)$, where M is running over all determining systems in a family of sets $\mathfrak{M}$, will be denoted by $\mathfrak{a}(\mathfrak{M})$. One can prove

(5) $$\mathfrak{a}(\mathfrak{a}(\mathfrak{M})) = \mathfrak{a}(\mathfrak{M}).$$

The operation leading from M to $\mathfrak{a}(M)$ is said to be the operation $(\mathfrak{a})$ or Souslin's operation. The sets from $\mathfrak{a}(\mathfrak{M})$ will be called $\mathfrak{M}$-Souslin sets. A determining system M is said to be regular if

$$M(i_1, \ldots, i_n) \supset M(i_1, \ldots, i_n, i_{n-1}).$$

In general the equality

(6) $$\mathfrak{a}(M) = \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} M(s),$$

is not true. The following condition (7) is sufficient, for (6) should be true.

$$(7) \qquad s, s' \in S_n \Rightarrow M(s) \cap M(s') = \varnothing .$$

Definition 2. Let M be a determining system in a space X, that means, in the family of all subsets of X. A M-Cauchy family is a centred family $\mathfrak{B}$ of subsets of X such that every $M(s)$, $s \prec \sigma$, contains a set from $\mathfrak{B}$ for some $\sigma \in \Sigma$. A determining system in a space X will be called complete if the intersection of closures of sets of any M-Cauchy family is non-void.

Definition 3. An analytic structure in a space X is a complete determining system M in X such that $\mathfrak{a}(M) = X$.

One can prove that analytic structures are invariant under continuous mappings (that means, if f is a continuous mapping of a space X onto a space Y and if M is an analytic structure in X, then $\{f[M(s)]\}$ is an analytic structure in Y).

Theorem 6. *A space X is analytic if and only if there exists an analytic structure in X.*

Proof. First let F be an analytic structure in X and let all $F(s)$ be closed. For every $s \in S$ put

$$\Sigma(s) = \{\sigma; \quad \sigma \in \Sigma, \quad s \prec \sigma\}.$$

It is easy to see that $\{\Sigma(s)\}$ is an analytic structure in Σ. Consider the determining system

$$M = \{M(s)\} = \{F(s) \times \Sigma(s)\}$$

in the topological product Y of X and Σ. Let Z be the nucleus of M. It is easy to see that M is complete in Y, and consequently, $L = \{Z \cap M(s)\}$ is an analytic structure in Z. Since (6) is satisfied by $\Sigma(s)$ (of course, reading Σ instead of M), one can show that M, and, consequently, also L, satisfy (6). Thus, $\{L(s); s \in S_n\}_{n=1}^{\infty}$ is a complete sequence of closed countable coverings of Z, and by Theorem 1, Z is an E-space. Clearly, the projection of Z to X is X.

Since analytic structures are invariant under continuous mappings, to prove necessity it is sufficient to show that there exists an analytic structure in every E-space. Let $\{\mathfrak{F}_n\}$ be a complete sequence of countable coverings of a space X. Arranging the covering $\mathfrak{F}_n$ in a sequence $\{\mathfrak{F}_n^k\}_{k=1}^{\infty}$ and putting

$$M(i_1, \ldots, i_n) = \bigcap_{k=1}^{n} M_k^{i_k}$$

we obtain an analytic structure M in X.

Theorem 7. *A space X is analytic if and only if X is Souslin with respect to closed subsets of every $Y \supset X$.*

Proof. If M is an analytic structure in X and if $Y \supset X$, then

$$\mathfrak{a}(\overline{M(s)}^{Y}) = X.$$

To prove necessity one can show that every determining system consisting of compact sets is complete and if M is a complete regular determining system consisting

of closed sets and if $\mathfrak{a}(M) = X$, then $\{X \cap M(s)\}$ is an analytic structure in X.

Note. From the preceding theorem and (6) it follows at once that the family of all analytic subspaces of a space is invariant under operation of Souslin. Since countable unions and intersections are Souslin's operations, the images under continuous mappings of spaces which are F_σ or $F_{\sigma\delta}$ or ... in some of their compactifications are analytic.

THEOREM 8. *A regular determining system M in a space X is complete if and only if the sets*

$$M(\sigma) = \bigcap_{s \prec \sigma} \overline{M(s)} \tag{8}$$

are compact and if a $M(\sigma)$ is contained in some open set V, then some $\overline{M(s)}$, $s \prec \sigma$, is contained in V.

Proof. Obviously, the condition is necessary. Conversely, let $\mathfrak{B}$ be a M-Cauchy family. We can assume that $M(s) \in \mathfrak{B}$ for all s, $s \prec \sigma$, where σ is an element of Σ. From the condition it follows at once that the family consisting of all sets of the form $\overline{B} \cap M(\sigma)$ is centred. Since $M(\sigma)$ is compact, the intersection of $\overline{\mathfrak{B}}$ is non-void.

THEOREM 9. *A space X is analytic if and only if there exists a mapping f of Σ to the family of all compact subspaces of X such that*

$$\bigcup_{\sigma \in \Sigma} f(\sigma) = X$$

and the set $\{\sigma : f(\sigma) \subset V\}$ is open in Σ for every open subset V of X.

Proof. By Theorem 8 the condition is necessary (put $f(\sigma) = M(\sigma)$). Conversely, let f be the mapping from the condition. Put

$$M(s) = \bigcup_{\sigma \succ s} f(\sigma).$$

Clearly, M is a regular determining system in X and $\mathfrak{a}(M) = X$. Let $\mathfrak{B}$ be a M-Cauchy family. We can assume that $M(s) \in \mathfrak{B}$ for all $s \prec \sigma$, where σ is an element of Σ. From the condition it follows that the family of all sets $f(\sigma) \cap B$, $\overline{B} \in \mathfrak{B}$, is centred.

THEOREM 10. *A space X is the image under a continuous mapping of the space of all irrational numbers of the unit interval of real numbers (that is, of Σ) if and only if there exist an analytic structure M in X such that the sets $M(\sigma)$ defined by (7) are one-point.*

Proof. Let us suppose the condition. Let $f(\sigma)$ be the point of $M(\sigma)$. By Theorem 8 the mapping f is continuous. Conversely, if f is a continuous mapping of Σ onto X, then $M = \{f[\Sigma(s)]\}$ is an analytic structure in X and from Theorem 8 it follows at once that the sets $M(\sigma)$ are one-point.

Note. Analytic spaces were introduced by G. Choquet in [2] and studied in [1] and [6]. For the sake of completeness let us note that a metrizable space X is analytic if and only if X is analytic in the classical sense, that means, X is the image under a continuous mapping of Σ. This result of G. Choquet and M. Sion follows from Theorem 7 (one must use a well known theorem) or directly from Theorem 10

Problem. *Is the class $\mathcal{E}$ P-closed*?

Problem 2. *To describe the class of all spaces X which are $F_{\sigma\delta}$ in every space $Y \supset X$.*

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MATHEMATICS

An Iterative Method for the Eigenproblem of Matrices

by
M. ALTMAN

Presented by W. ORLICZ on July 26, 1391

This paper contains an iterative method for computing eigenvalues and eigenvectors of real symmetric matrices. The method presented here is based on a general method of finding zero elements of a non-linear functional in a Banach space (see [1]). However, the proof of the convergence of the method in question does not follow from the general theory developed in [1] and [2] and requires an independent argument. The advantage of this method is that starting with an arbitrary vector we get an eigenvalue and the corresponding eigenvector. Thus, the approximate eigenvalue and the corresponding eigenvector obtained by this method depend on the starting vector. Hence, we obtain generally different eigenvalues and corresponding eigenvectors by starting with different initial vectors. The method is suitable for programming on computer. A method based on the same idea is contained in [3].

Let A be a real and symmetric matrix of order N. Thus, A can be considered as a linear operator with domain and range in the Euclidean space R_N.

The problem is to find eigenvalues λ and corresponding eigenvectors $x \in R_N$ satisfying the equation

$$Ax = \lambda x, \quad x \neq 0. \tag{1}$$

It follows from (1) that

$$\lambda(x) = \frac{(Ax, x)}{(x, x)}. \tag{2}$$

Hence, we get, by (2),

$$Ax = \lambda(x)\, x, \quad x \neq 0.$$

Instead of Eq. (1) we shall solve the following functional equation

$$F(x) = \|Ax - \lambda(x)\, x\|^2 = 0, \quad x \neq 0. \tag{3}$$

Thus, we see that our eigenproblem is equivalent to that of finding the solutions of the functional equation (3). But Eq. (3) is a non-linear functional equation with a functional $F(x)$ being continuously differentiable in the sense of Fréchet. A method of solving such non-linear functional equations was given in [1], [2].

We shall now apply the method described in [1] to solve Eq. (3). For this purpose let us calculate the Fréchet derivative $F'(x)$ of the functional $F(x)$ defined by (3). We have

$$\tfrac{1}{2} F'(x)\, u = \big(Ax - \lambda(x)\, x, \quad Au - \lambda(x)\, u - \lambda'(x)(u)\, x\big),$$

where the linear functional $\lambda'(x)$ for fixed x denotes the Fréchet derivative of the functional $\lambda(x)$ defined by (2). Since

$$\big(Ax - \lambda(x)\, x, \quad \alpha'(x)(u)\, x\big) = 0,$$

we get

$$\tfrac{1}{2} F'(x)\, u = \big(Ax - \lambda(x)\, x, \quad Au - \lambda(x)\, u\big).$$

Hence, we obtain

(4) $$y = \tfrac{1}{2} F'(x) = (A - \lambda(x)\, I)^2\, x,$$

where I denotes the identity mapping.

Let x_0 be the initial approximate solution of Eq. (3). Then, following [1], the sequence of approximate solutions x_n of Eq. (3) is defined as follows:

(5) $$x_{n+1} = x_n - \frac{F(x_n)}{\|F'(x_n)\|^2} F'(x_n).$$

In virtue of (4) we get, by (5),

(6) $$x_{n+1} = x_n - \frac{F(x_n)}{2\,\|y_n\|^2} y_n,$$

where

$$y_n = (A - \lambda(x_n)\, I)^2\, x_n.$$

A simple calculation gives, by (2),

(7) $$x_{n+1} = x_n - \frac{\big\|\,\|x_n\|^2 A x_n - (Ax_n, x_n)\, x_n\big\|^2}{2\,\|z_n\|^2} z_n,$$

where

$$z_n = (\|x_n\|^2 A - (Ax_n, x_n)\, I)^2\, x_n.$$

We shall now establish some relations needed in the sequel. First of all let us observe that

(8) $$(y, x) = F(x).$$

It follows from (8) and (5) that

(9) $$\|x_{n+1}\|^2 = \|x_n\|^2 - \tfrac{3}{4} \frac{F^2(x_n)}{\|y_n\|^2}.$$

It follows from (9) that the sequence of $\|x_n\|^2$ is decreasing and bounded. Hence, it is convergent.

Relation (9) yields

(10) $$\|x_{n+1}\|^2 = \|x_0\|^2 - \tfrac{3}{4}\sum_{i=0}^{n}\frac{F^2(x_i)}{\|y_i\|^2}.$$

Hence, we get the convergence of the series

(11) $$\sum_{i=0}^{\infty}\frac{F^2(x_i)}{\|y_i\|^2} < \infty.$$

It follows from (10) that

$$x_n \to 0 \quad \text{as} \quad n \to \infty$$

if and only if

(12) $$\|x_0\|^2 = \tfrac{3}{4}\sum_{i=0}^{\infty}\frac{F^2(x_i)}{\|y_i\|^2}.$$

We shall now prove the convergence of the sequence of numbers (Ax_n, x_n). We have, by (5),

$$(Ax_{n+1}, x_{n+1}) = \left(Ax_n - \frac{F(x_n)}{2\|y_n\|^2}Ay_n,\ x_n - \frac{F(x_n)}{2\|y_n\|^2}y_n\right).$$

Hence, we get

(13) $$(Ax_{n+1}, x_{n+1}) - (Ax_n, x_n) = \frac{F^2(x_n)}{4\|y_n\|^2}\left[\frac{(Ay_n, y_n)}{\|y_n\|^2} - 4\frac{(Ay_n, x_n)}{F(x_n)}\right].$$

We shall show that the expression in square brackets in (12) is bounded. In fact, we have

(14) $$\frac{|(Ay_n, y_n)|}{\|y_n\|^2} \leqslant M \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

where

(15) $$M = \max_{1 \leqslant i \leqslant N} |\lambda_i|.$$

Further, we have, by (4),

$$(Ay, x) = \big((A - \lambda(x) I)^2 x, Ax\big) = \big((A - \lambda(x) I) x, A(A - \lambda(x) I) x\big).$$

Hence, we get, (15) and (8),

$$|(Ay, x)| \leqslant 2MF(x)$$

or

(16) $$\frac{|(Ay_n, x_n)|}{F(x_n)} \leqslant 2M \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

If $\|y_n\|$ in (14) or $F(x_n)$ in (16) vanishes, then, of course, x_n is an eigenvector. Thus, the boundedness of the expression in square brackets in (13) follows from (14) and (16).

It follows from (13) that the convergence of the series in (11) implies the same for the following series

$$\sum_{n=0}^{\infty} (Ax_{n+1}, x_{n+1}) - (Ax_n, x_n) = \lim_{k\to\infty} (Ax_k, x_k) - (Ax_0, x_0).$$

Thus, we have proved the convergence of the sequence of numbers (Ax_n, x_n).

Suppose now that the vector x_0 is so chosen that condition (12) is not satisfied. Then the sequence of $\|x_n\|^2$ converges and its limit is different from zero. Hence, it follows that the sequence

$$\left\{\frac{(Ax_n, x_n)}{(x_n, x_n)}\right\}, \qquad n = 0, 1, 2, \ldots$$

is convergent. Denote by λ its limit. It is easy to see that λ is an eigenvalue of A. In fact, it follows from (9) because of the boundedness of the sequence of $\|y_n\|^2$ that

$$F(x_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{17}$$

There is a subsequence of $\{x_n\}$ convergent to some element x^* different from the zero-element. In virtue of (17) and (3) we get

$$\|Ax^* - \lambda x^*\| = 0.$$

Hence, x^* is the eigenvector corresponding to the eigenvalue λ. It is obvious that if the eigenvalue λ is simple, then we have

$$x_n \to x^* \quad \text{as} \quad n \to \infty.$$

We shall now give an estimate for the approximate eigenvectors x_n defined by process in (5). Let $e_1, e_2, \ldots, e_N$ be the normed eigenvectors of the matrix A. They form an orthogonal system and any vector x of R_N can be represented in the form

$$x = \sum_{i=1}^{N} \xi_i e_i.$$

In virtue of (3) we get the following expression for $F(x)$.

$$F(x) = \sum_{i=1}^{N} ((\lambda_i - \lambda)(x)^2 \xi_i^2,) \tag{18}$$

where λ_i are the eigenvalues corresponding to e_i. Suppose that the multiplicity of the eigenvalue $\lambda = \lambda_i$ is $r < N$. Let $e_1, e_2, \ldots, e_r$ be the eigenvectors corresponding to λ. Denote by c the distance from λ to the nearest eigenvalue of A. For sufficiently large n we have the following inequalities

$$\left|\lambda_i - \frac{(Ax_n, x_n)}{(x_n, x_n)}\right| \geqslant \frac{c}{2} \qquad \text{for} \qquad i = r+1, r+2, \ldots, N. \tag{19}$$

Put

$$x_n^* = \sum_{i=1}^{r} \xi_{in} e_i, \tag{20}$$

where

$$x_n = \sum_{i=1}^{N} \xi_{in} e_i. \tag{20'}$$

Of course, x_n^* is an eigenvector corresponding to λ and we have

$$\|x_n^* - x_n\|^2 = \sum_{i=r+1}^{N} \xi_{in}^2. \tag{21}$$

It follows from (18) and (19) that

$$F(x_n) \geqslant \frac{c^2}{2} \sum_{i=r+1}^{N} \xi_{in}^2. \tag{22}$$

Relations (22) and (21) imply

$$\|x_n^* - x_n\| \leqslant \frac{\sqrt{2F(x_n)}}{c}. \tag{23}$$

We shall now give an error estimate for the approximate eigenvalue. We have, by (20′),

$$\sum_{i=1}^{r} \xi_{in}^2 = \|x_n\|^2 - \sum_{i=r+1}^{N} \xi_{in}^2.$$

Relation (22) implies

$$\sum_{i=r+1}^{N} \xi_{in}^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, for sufficiently large n we get

$$\sum_{i=1}^{r} \xi_{in}^2 \geqslant \frac{\|x_n\|^2}{2}. \tag{24}$$

It follows from (24) and (18) that

$$F(x_n) \geqslant \tfrac{1}{2} \|x_n\|^2 \left(\lambda - \frac{(Ax_n, x_n)}{(x_n, x_n)}\right)^2.$$

Hence, we get

$$\left|\lambda - \frac{(Ax_n, x_n)}{(x_n, x_n)}\right| \leqslant \frac{\sqrt{2F(x_n)}}{\|x_n\|}. \tag{25}$$

If λ is a simple eigenvalue of A then we get, instead of (23),

$$\left\|\frac{(x_n, x^*)}{\|x^*\|^2} x^* - x_n\right\| \leqslant \frac{\sqrt{2F(x_n)}}{c}, \tag{26}$$

where x^* is the limit of the sequence of x_n.

Thus, we have proved the following

THEOREM. *Let* x_0 *be an arbitrary vector such that condition* (12) *is not satisfied. Then the sequence of numbers* $\frac{(Ax_n, x_n)}{(x_n, x_n)}$ *converges to an eigenvalue* λ *of the matrix* A.

If λ *is a simple eigenvalue of* A *then the sequence of approximate eigenvectors* x_n *defined by the process in* (6) *or* (7) *converges to the eigenvector* x^* *corresponding to* λ *and the error estimate is given by the formula* (26) *for sufficiently large* n. *If the eigenvalue* λ *has the multiplicity greater than one, then there exists an eigenvector* x_n^* *corresponding to* λ *such than the relation* (23) *is satisfied for sufficiently large* n.

Remark. It is rather unlikely that condition (12) may be satisfied in practice. Thus, the process defined bv (6) or (7) practically always gives an approximate eigenvalue of A and a corresponding approximate eigenvector.

The same method can be applied to linear operators in Hilbert space.

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MATHEMATICS

A Generalized Method of Steepest Descent

by

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Presented by W. ORLICZ on July 26, 1391

1. Paper [1] contains an iterative method for the approximate solutions of linear equations in Hilbert space. This is actually a kind of method of steepest descent. The main idea thereof is to introduce an auxiliary subspace so as to improve the convergence of the iterative process. Thus, we have a kind of over-relaxation depending on a subspace as a parameter. The auxiliary subspace can be introduced in many ways, there is however one necessary condition which must be satisfied, namely the auxiliary subspace should be orthogonal to the vector b of the equation in (1). The simplest case when the auxiliary subspace consists only of the zero element of the space gives the method of steepest orthogonal descent considered in [2].

This paper follows the same general idea of introducing an auxiliary subspace to improve the convergence of the iterative process. This can be done in many ways but we have no restriction of orthogonality of the auxiliary subspace. By a suitable choice of the auxiliary subspace the convergence of the method can be much faster than that of the steepest descent investigated by L.V. Kantorovitch [3]. In the case when the auxiliary subspace consists only of the zero element of the space, we obtain again the method of steepest orthogonal descent as in [2].

An application of the method to the approximate solution of a system of linear algebraic equation is also given.

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. Consider the linear equation

$$Ax = b, \quad x, b \in H. \tag{1}$$

Let us consider the case when A is selfadjoint and positive definite, or, more precisely,

$$A^* = A,$$

where A^* is the adjoint of A, and

$$m(x, x) \leqslant (Ax, x) \leqslant M(x, x), \tag{2}$$

where $0 < m < M < \infty$, m and M being the minimum and maximum eigenvalues of A, respectively.

The general case when A is a non-singular bounded linear operator can be reduced to that mentioned above. We shall first consider the following case.

Let L be an arbitrary subspace of H orthogonal to b, and put

(3) $$Ł = A(L), \quad L \perp b$$

i.e. $Ł$ is the image of the subspace L. $Ł$ is the auxiliary subspace mentioned above.

Denote by $P_Ł\, x$ the orthogonal projection of x onto the subspace $Ł$. Let us put

(4) $$b' = b - P_Ł\, b.$$

If

(5) $$P_Ł\, b = A\tilde{b}$$

then we have, by (4),

(6) $$b' = b - A\tilde{b}.$$

Instead of Eq. (1) consider now the following equation

(7) $$Ax = b'.$$

Thus, if x^* is the solution of Eq. (1) and x' is the solution of Eq. (7), then we have, by (6),

(8) $$x^* = x' + \tilde{b}.$$

Denote by $P_{b'}\, x$ the orthogonal projection of x onto the one dimensional subspace containing b' defined by (4). Let Z be the complementary subspace of H orthogonal to b' and to $Ł$. Denote by $P_Z\, x$ the orthogonal projection of x onto the subspace Z. We have

(9) $$P_Z x = x - P_{b'}\, x - P_Ł\, x.$$

Since we have assumed that

$$Z \perp b' \quad \text{and} \quad Z \perp Ł,$$

we get, by (4),

(10) $$Z \perp b.$$

Following the argument of [1] let us now define the linear operator R in this way

$$z = Rx = P_Z Ax = Ax - P_{b'}\, Ax - P_Ł\, Ax,$$

where the expression of $P_Z x$ is given in (9). Hence, we get the following relation

(11) $$z = Rx = Ax - \frac{(Ax, b')}{(b', b')}\, b' - A\tilde{x},$$

where

$$A\tilde{x} = P_Ł\, Ax.$$

Consider now the following homogeneous equation

(12) $$Rx = 0, \quad x \neq 0,$$

where operator R is as defined by (11).

We shall now define an iterative process for the approximate solutions of Eq. (12). Let x_0 be an arbitrary element of the space H satisfying the following condition

(13) $$(x_0, b) \neq 0.$$

The sequence of approximate solutions x_n of Eq. (12) is defined as follows

(14) $$x_{n+1} = x_n - a_n z_n; \quad n = 0, 1, 2, \ldots,$$

where

(15) $$z_n = Rx_n.$$

Hence, it follows that

(16) $$z_{n+1} = z_n - a_n Rz_n .$$

We shall now determine the coefficients a_n in (14) using the following quadratic functional

(17) $$F(z) = (R^{-1} z, z)$$

as an error measure, where $F(z)$ is defined for all z of Z. To show that this definition is correct let us consider operator R defined only on the subspace Z. It is easy to see that the following relation is true

(18) $$(Rz, z) = (Az, z)$$

for any z of Z. Hence, it follows that operator R is positive definite and selfadjoint in the subspace Z. Thus, we proved the existence of the inverse R^{-1} defined on the whole of Z. We shall now choose a_n in (14) and in (16) so as to minimize $F(z_{n+1})$, where $F(z)$ is defined in (17).

Thus, we get, by (18),

(19) $$a_n = \frac{(z_n, z_n)}{(Rz_n, z_n)} = \frac{(z_n, z_n)}{(Az_n, z_n)},$$

and for the approximate solutions of Eq. (12) we get the following relations instead of (14) and (16)

(20) $$x_{n+1} = x_n - \frac{(z_n, z_n)}{(Rz_n, z_n)} z_n ,$$

(21) $$z_{n+1} = z_n - \frac{(z_n, z_n)}{(Rz_n, z_n)} Rz_n .$$

where operator R is as defined by (11).

Denote by m_Z and M_Z the minimum and maximum eigenvalues of operator R with domain and range in Z. Thus, we have

(22) $$m_Z(z, z) \leqslant (Rz, z) \leqslant M_Z(z, z),$$

where

(23) $$m \leqslant m_Z \quad \text{and} \quad M_Z \leqslant M$$

in virtue of (18). Hence, it follows that

(24) $$M_Z^{-1} (z, z) \leqslant (R^{-1} z, z) \leqslant m_Z^{-1} (z, z).$$

Following the same argument as in [1] we get

$$\frac{F(z_{n+1})}{F(z_n)} \leqslant q_Z^2, \tag{25}$$

where

$$q_Z = \frac{M_Z - m_Z}{M_Z + m_Z}. \tag{26}$$

Relations (24)—(26) yield

$$\|z_n\| \leqslant \left(\frac{M_Z}{m_Z}\right)^{1/2} \|z_0\|\, q_Z^n. \tag{27}$$

Thus, it follows that the sequence of z_n converges to the zero element of H at least as fast as a geometric progression with the ratio q_Z. Using the same argument as in [1] we can prove that the sequence of x_n converges to a solution x of Eq. (12) and we have

$$x_n \to x = x_0 - z, \tag{27'}$$

where z is an element of Z.

If follows from (10) in the same way as in [1] that x is a non-trivial solution of Eq. (12).

Now, if for x_n defined by formula (20) the norm of the corresponding z_n is sufficiently small, then we calculate the approximate solution y_n' of Eq. (7) by the following formula

$$y_n' = \frac{\|b'\|^2}{(Ax_n, b')}(x_n - \tilde{x}_n), \tag{28}$$

where

$$P_L A x_n = A\tilde{x}_n.$$

Hence, it follows in virtue of (4)—(8) that the approximate solution y_n of Eq. (1) is given by the following formula

$$y_n = y_n' + \tilde{b},$$

where y_n' and $\tilde{b}$ are defined by (28) and (5), respectively.

Arguing as in [1] one can prove that

$$(Ax_n, b') \to (Ax, b') \neq 0 \quad \text{as} \quad n \to \infty, \tag{29}$$

where x is the same as in (27′). In virtue of (27) using the same argument as in [1] we get the following error estimate for the approximate solution of Eq. (7):

$$\|y_n' - x'\| \leqslant \frac{\|b'\|^2\,\|z_0\|}{m|(Ax_n, b')|}\left(\frac{M_Z}{m_Z}\right)^{1/2} q_Z^n,$$

where x' is the solution of Eq. (7) and q_Z is as defined by (26).

2. We shall now consider the general case. In this case subspace L is not supposed to be orthogonal to b and it can be chosen arbitrarily. However, the following condition should be satisfied.

The subspace spanned by the elements of L and of Z is not the whole of H, Z being defined in par. 1. In other words, there exists an element $v \in H$ such that

(30) $$v \perp L \quad \text{and} \quad v \perp Z.$$

Condition (30) is satisfied if L is a finite dimensional subspace of H. This case is very important for the application to the solution of a system of linear algebraic equations.

All assertions of par. 1 remain also true in the general case. However, we should start the iteration process in (20) with a vector x_0 satisfying the following condition instead of (13)

(31) $$(x_0, v) \neq 0.$$

Using the same argument as in par. 1 one can prove that the sequence of the approximate solutions x_n defined in (20) converges to a non-trivial solution of Eq. (12). In the same way as in par. 1 it follows from (30) and (31) that condition (29) is also satisfied in general case.

Remark 1. It is rather unlikely that condition (31) will not be satisfied in practice. Thus, the process defined in (20) gives practically always an approximate solution of Eq. (12).

3. In the same way as in [1] one can introduce an additional vector as a parameter. This can be done in two ways. Let y be an arbitrary vector of H. Then in the first case we introduce the parameter by considering the equation

$$A(x+y) = Ay+b$$

instead of Eq. (1). In this way we introduced the vector parameter in [1].

In the second case we consider the equation

$$A(x+y') = Ay'+b'$$

instead of Eq. (7). Thus, we see that we can even introduce two vector parameters independently.

Remark 2. If operator A is defined by a matrix (N by N) then operator R can be written in a matrix form exactly in the same way as in [1].

Various methods can be obtained depending on the construction of the subspace L. Some of them will be given later.

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MATHEMATICS

A General Method with Minimum Ortho-Residua

by

M. ALTMAN

Presented by W. ORLICZ on July 20, 1391

1. Paper [1] contains an iterative method of solving linear equations in Hilbert space, which gives, in particular, an iterative method for the solution of a system of linear algebraic equations. The method presented there consists in replacing the given equation by a homogeneous one which is then solved by the method with minimum residua described by M. A. Krasnoselsky and S. G. Krein [2]. The characteristic feature of the method in [1] is that the new residua, i.e. the residua of the corresponding homogeneous equation, are orthogonal to the free vector b of the equation (1). Besides, the speed of the convergence of this method depends also on the vector b. This fact makes it possible to introduce an arbitrary vector as a parameter. Thus, changing the parameter vector, we change the speed of the convergence at the same time. From this point of view the method presented in [1] gives a kind of over-relaxation, where the numerical parameter is replaced by an arbitrary vector.

In this paper we give a generalization of the main idea contained in [1]. Introducing an auxiliary subspace we obtain that the residua of the homogeneous equation belong to a subspace orthogonal to vector b. This fact improves the convergence of the method. The auxiliary subspace can be introduced in many ways. The speed of the convergence depends on two factors: on b and on the choice of the auxiliary subspace. Thus, we obtain a kind of over-relaxation depending on a subspace as a parameter. The general method with minimum orthogonal residua presented here is faster than the method in [1] and by a suitable choice of the auxiliary subspace it can be much faster than the method with minimum residua investigated by Krasnoselsky and Krein [2]. An application to the solution of a system of linear algebraic equations is also given.

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. Consider the linear equation

(1) $$Ax = b, \quad x, b \in H.$$

We shall assume operator A to be selfadjoint and positive definite, or, more precisely,

$$A^* = A,$$

where A^* is the adjoint of A, and

$$m(x, x) \leqslant (Ax, x) \leqslant M(x, x),$$

where $0 < m < M < \infty$, m and M being the minimum and maximum eigenvalues of A, respectively.

Without loss of generality one can suppose that $\|b\| = 1$. Now, let L be an arbitrary subspace of H orthogonal to b and to Ab, i.e.

$$L \perp b \quad \text{and} \quad L \perp Ab.$$

Let us put

$$Ł = A(L),$$

i.e. $Ł$ is the image of the subspace L. Then the subspace $Ł$ is orthogonal to b. $Ł$ is the auxiliary subspace mentioned above.

Let Z be the complementary subspace of H orthogonal to b and to $Ł$, i.e.

$$Z \perp b \quad \text{and} \quad Z \perp Ł.$$

Denote by $P_b x$ the orthogonal projection of $x \in H$ onto the subspace of H spanned by the vector b.

Denote by $P_Ł x$, $P_Z x$ the orthogonal projection of x onto the subspace $Ł$, Z of H, respectively.

We have

$$P_Z x = x - P_b x - P_Ł x. \tag{2}$$

Let us now define the following linear operator R

$$z = Rx = P_Z Ax = Ax - P_b Ax - P_Ł Ax.$$

Hence, we get the following expression for Rx

$$z = Rx = Ax - (Ax, b) b - A\tilde{x}, \tag{3}$$

where

$$A\tilde{x} = P_Ł Ax.$$

Consider the following homogeneous equation

$$Rx = 0, \quad x \neq 0, \tag{4}$$

where operator R is defined by (3).

We shall now define an iterative process of solving Eq. (4). Let x_0 be an arbitrary element of H satisfying the following condition

$$(x_0, b) \neq 0. \tag{5}$$

The sequence of approximate solutions x_n of Eq. (4) is defined as follows

$$x_{n+1} = x_n - \alpha_n z_n, \quad n = 0, 1, 2, \ldots \tag{6}$$

where

$$z_n = Rx_n. \tag{6'}$$

Thus, we have

$$z_{n+1} = z_n - \alpha_n Rz_n. \tag{7}$$

To determine the coefficients a_n in (6) we proceed as in the method with minimum residua [2], i.e. the coefficients are so chosen as to minimize the norm of z_{n+1}.

Thus, we obtain by a simple argument that

$$a_n = \frac{(Rz_n, z_n)}{\|Rz_n\|^2} \tag{8}$$

and the iterative process defined by (6) and (7) is of the form

$$x_{n+1} = x_n - \frac{(Rz_n, z_n)}{\|Rz_n\|^2} z_n . \tag{9}$$

$$z_{n+1} = z_n - \frac{(Rz_n, z_n)}{\|Rz_n\|^2} Rz_n . \tag{10}$$

where operator R is as defined by Eq. (3).

We shall now examine the process defined in (9), (10). For this purpose let us consider an operator R defined only on the subspace Z. It is easy to see that the operator R with domain and range in Z is selfadjoint and positive definite, since the following relation is true:

$$(Rz, z) = (Az, z) \tag{11}$$

for any z of Z.

Denote by m_Z, M_Z the minimum and maximum eigenvalues of R in Z, respectively.

Thus, we have

$$m_Z (z, z) \leqslant (Rz, z) \leqslant M_Z (z, z), \tag{12}$$

whence, in virtue of (11), we get

$$m \leqslant m_Z, \quad M_Z \leqslant M. \tag{12'}$$

We shall show that the sequence of z_n defined by (10) converges to the zero element of H.

We have

$$\|z_{n+1}\|^2 = \|z_n\|^2 - \frac{(Rz_n, z_n)^2}{\|Rz_n\|^2} .$$

Hence, it follows that

$$\frac{\|z_{n+1}\|^2}{\|z_n\|^2} = 1 - \frac{(Rz_n, z_n)^2}{\|z_n\|^2 \, \|Rz_n\|^2} . \tag{13}$$

Since (see [1])

$$\frac{(Rz_n, z_n)^2}{\|z_n\|^2 \, \|Rz_n\|^2} \geqslant \frac{4m_Z M_Z}{(M_Z+m_Z)^2} ,$$

we get

$$\frac{\|z_{n+1}\|}{\|z_n\|} \leqslant q_Z . \tag{14}$$

where

$$q_Z = \frac{M_Z - m_Z}{M_Z + m_Z} , \tag{15}$$

Thus, the sequence of z_n converges to zero at least as fast as a geometric progression with ratio q_Z defined by (15). Relation (13) yields, by (11),

$$\frac{\|z_{n+1}\|^2}{\|z_n\|^2} = 1 - \frac{(Az_n, z_n)^2}{\|z_n\|^2\,\|Az_n\|^2} \cdot \frac{\|Az_n\|^2}{\|Rz_n\|^2}.$$

Hence, we get, by (2) and (3),

$$\frac{\|z_{n+1}\|^2}{\|z_n\|^2} \leqslant 1 - \frac{4mM}{(M+m)^2} \cdot \frac{\|Az_n\|^2}{\|Az_n\|^2 - (Az_n, b)^2 - \|P_L\, Az_n\|^2}.$$

This relation shows immediately that the method defined by (9), (10) is faster than the method in [1] and that both methods are faster than the method with minimum residua in [2].

We shall show that the sequence of x_n defined by (9) converges to a non-trivial solution of Eq. (4). It follows from (9) that

$$x_n = x_0 - a_0 z_0 - \ldots - a_{n-1} z_{n-1}, \tag{16}$$

where $z_i \in Z$ for $i = 0, 1, 2, \ldots$.

Hence, we infer, by (14), (15) and (8), that the sequence of x_n is convergent. Consequently, we have

$$x_n \to x = x_0 - z \quad \text{as} \quad n \to \infty, \tag{16'}$$

where z is an element of Z. It is easy to see that $x = x_0 - z$ is a solution of Eq. [4]. In fact, we have, by (6′),

$$z_n = Rx_n \to Rx.$$

Since the sequence of z_n converges to zero, we get $Rx = 0$. It remains to show that x is a non-trivial solution of Eq. (4). We have, by (16) and (5),

$$(x_n, b) = (x_0, b) \neq 0,$$

since the elements z_i are orthogonal to b.

Hence, we get

$$\|x_n\| \geqslant |(x_0, b)| > 0.$$

Suppose now that for x_n defined by (9) the norm of the corresponding z_n is sufficiently small. Then we calculate the approximate solution y_n of Eq. (1) by the following formula

$$y_n = \frac{1}{(Ax_n, b)}(x_n - \widetilde{x}_n),$$

where $\widetilde{x}_n$ is defined by the relation

$$P_L\, Ax_n = A\widetilde{x}_n. \tag{17}$$

It is easy to see that

$$(Ax_n, b) \to (Ax, b) \neq 0.$$

In fact we have, by (3),

$$z_n = Rx_n = A(x_n - \widetilde{x}_n) - (Ax_n, b)\, b, \tag{18}$$

where $\widetilde{x}_n$ is defined by (17).

It follows from (17) that

$$\tilde{x}_n \to \tilde{x} \quad \text{as} \quad n \to \infty,$$

where $P_L Ax = A\tilde{x}$, x being the solution of Eq. (4) determined by relation (16′). Let us observe that $A\tilde{x}$ is in L, or, equivalently, $\tilde{x}$ is in L. Thus, we obtain from (18) the following equation which is equivalent to Eq. (4).

$$Rx = A(x - \tilde{x}) - (A\tilde{x}, b)\, b = 0.$$

If we now suppose that $(Ax, b) = 0$, then we should get, by (16′),

$$x = x_0 - z = \tilde{x},$$

where $z \in Z$ and $\tilde{x} \in L$.

Hence, we get the following relation

$$(x_0, b) = 0,$$

which is a contradiction to (5).

We shall now give an error estimate for the approximate solution y_n of Eq. (1). Let x^* be the solution of Eq. (1). Then we have, by (18), (14) and (12),

$$\|y_n - x^*\| \leqslant \frac{\|z_n\|}{m\,|(Ax_n, b)|} \leqslant \frac{\|z_0\|}{m\,(x_n, Ab)}\, q_Z^n, \tag{19}$$

where q_Z is as defined by (15).

Thus, we see in virtue of (19) and (12′) that by a suitable choice of the auxiliary subspace the general method with minimum orthogonal residua can be much faster than the method with minimum residua investigated by Krasnoselsky and Krein [2].

One can obtain various methods by different choice of the subspace L. In particular, if we replace L by the subspace consisting of the zero element only, then we obtain the method given in [1].

2. In the same way as in [1] one can introduce an arbitrary vector as an additional parameter.

Let y be an arbitrary vector of H. If x^* is the solution of Eq. (1), then, obviously, we have

$$A(x^* + y) = Ay + b.$$

Putting

$$b' = \frac{Ay + b}{\|Ay + b\|}, \qquad x = \frac{x^* + y}{\|Ay + b\|},$$

we obtain the following equation

$$Ax = b' \tag{20}$$

instead of Eq. (1).

To solve Eq. (20) we can apply the method described in par. 1. Then the subspace L should be orthogonal to b' and to Ab'. We start the iterative process with an arbitrary vector x_0 which is not orthogonal to b'. Instead of operator R we define now the operator R_y replacing in (3) b by b'. Thus, we obtain

$$R_y x = Ax - \frac{(Ax, Ay + b)}{\|Ay + b\|^2}(Ay + b) - A\tilde{x} = 0, \tag{21}$$

where

$$A\widetilde{x} = P_{Ł}^{\cdot} Ax.$$

Replacing R by R_y in (9), (10) we obtain the iterative process for solving the Eq. (21). Thus, the approximate solution of Eq. (1) is given by the following formula

$$y_n = \frac{\|Ay+b\|^2}{(Ax_n, Ay+b)}(x_n - \widetilde{x}_n) - y,$$

where $\widetilde{x}_n$ is as defined in (17). Let Z_y be the complementary subspace of H orthogonal to b' and to $Ł = A(L)$. Then operator R_y with domain and range in Z_y is selfadjoint and positive definite having m_y and M_y as its minimum and maximum eigenvalues. respectively. The error estimate is given by the following formula

$$\|y_n - x^*\| \leqslant \frac{\|Ay+b\|^2 \|z_0\|}{m\,|(Ax_n, Ay+b)|} q_y^n,$$

where

$$q_y = \frac{M_y - m_y}{M_y + m_y}.$$

Remark. Suppose that the operator A is defined by the matrix

$$A = (a_{ik}), \quad i, k = 1, 2, \ldots, N.$$

Let $l_1, l_2, \ldots, l_s$ be linearly independent vectors orthogonal to b and Ab. Denote by $u_1, u_2, \ldots, u_s$ $(s < N)$ the orthonormal system obtained from $Al_1, Al_2, \ldots, Al_s$ by an orthogonalization process. This orthonormal system forms a basis in the subspace $Ł$ orthogonal to b. Then operator R can be given in the following matrix form.

Put

$$b = (b_1, b_2, \ldots, b_N), \quad c_k = (a_k, b), \quad a_k = (a_{1k}, a_{2k}, \ldots, a_{Nk}),$$

$$l_j = (l_{1j}, l_{2j}, \ldots, l_{Nj}), \quad d_{kt} = (a_k, l_j), \quad j = 1, 2, \ldots, S.$$

Then we have

$$R = (r_{ik}), \quad i, k = 1, 2, \ldots, N,$$

where

$$r_{ik} = a_{ik} - b_i c_k - l_{i1} d_{k1} - l_{i2} d_{k2} - \ldots - l_{is} d_{ks}.$$

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MATHEMATICS

A General Majorant Principle for Functional Equations

by

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Instead of operator equations in Banach spaces one can always solve functional equations in such spaces. In other words, one can reduce always solutions of operator equations to finding zero elements of non-linear functionals. Various iterative methods of higher order have been studied for solving functional equations in Banach spaces. Thus, a second order iterative method of solving such equations is contained in [1]. This method is actually a generalization of Newton's well-known classical method. Our abstract variant is essentially different from that given by L. V. Kantorovitch [8], [9]. A third order iterative method of solving functional equations is given in [4]. This method is a generalization of the well-known method of Tchebyshev for finding roots of a numerical function. Another class of iterative methods of higher order for finding zero elements of non-linear functionals in Banach spaces is considered in [3]. The formalism of these methods is based on the application of König's theorem. This class contains in particular Newton's method and an abstract variant of the method of tangent hyperbolas. Another iterative method of the third order is given in [7]. This method is an abstract generalization of the classical method of Laguerre. Namely, Laguerre has given an elegant method which applies only to algebraic equations having all real roots.

This paper contains a further contribution to iterative methods of higher order. We investigated all these methods mentioned above by using, among others, the majorant principle. However, this principle has been applied to each method individually. Thus, there arose the question of finding a general procedure of examining these methods. Such general majorant principle is presented in this paper. Actually we consider a class of methods which contains in particular Newton's method and also the methods of the third order quoted above. All these methods, except the method of Laguerre, have also abstract operator variants. However, the application of the operator variant of these methods presents very great practical difficulties. In fact, we need then the existence of the inverse of the Fréchet derivative and the estimate of its norm. In addition, at each iteration step we must solve a linear operator equation. This is the case of the operator variant of Newton's method. Operator variants of iterative methods of higher order are much harder to apply.

Let X be a Banach space and let $S(x_0, r)$ denote a closed sphere in X with centre x_0 and radius r. Let $F(x)$ be a non-linear continuous functional defined on the sphere $S(x_0, r)$. Suppose next that $F(x)$ is three times continuously differentiable in the sense of Fréchet in the sphere $S(x_0, r)$ and denote by $F'(x)$, $F''(x)$ and $F'''(x)$ the first, second and third derivatives of $F(x)$, respectively.

Consider the functional equation

$$F(x) = 0, \quad x \in X. \tag{1}$$

Let us consider a sequence of elements x_n of X and choose another sequence of elements y_n of X satisfying the following condition

$$\|y_n\| = 1, \quad F'(x_n)\, y_n = \|F'(x_n)\|, \quad n = 0, 1, 2, \ldots, \tag{2}$$

provided that such choice is possible. If the norm of the linear functional $F'(x_n)$ cannot be reached at the surface of the unit sphere, then it is enough to choose elements y_n of the unit sphere with values $F'(x_n)\, y_n$ sufficiently near to the norms of the corresponding functionals (see [6]).

Now weigh the real equation

$$f(z) = 0, \tag{3}$$

where $f(z)$ is real function of the real variable z, being three times continuously differentiable in the segment (z_0, z').

Following the argument of [2]—[4], [7] let us say that Eq. (1) possesses a real majorant Eq. (3), if the following conditions are satisfied

1° $\|F(x_0)\| \leqslant f(z_0); \quad z_0 \geqslant 0;$

2° $\dfrac{1}{\|F'(x_0)\|} \leqslant B_0, \quad B_0 = -\dfrac{1}{f'(z_0)} > 0;$

3° $\|F''(x)\| \leqslant f''(z)$ if $\|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0$.

4° $\|F'''(x)\| \leqslant f'''(z)$ if $\|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0$.

In the case of Newton's method the first three conditions are sufficient to define the majorant principle.

Consider now a class of iterative methods defined as follows. Let $g(z)$ be a real function of the real variable z. Put

$$A_n = \frac{F(x_n)\, F''(x_n)\, y_n^2}{[F'(x_n)\, y_n]^2}, \quad n = 0, 1, 2, \ldots \tag{4}$$

Then the sequence of the approximate solutions x_n of Eq. (1) is defined by the following formula

$$x_{n+1} = x_n - g(A_n)\, \frac{F(x_n)}{F'(x_n)\, y_n}\, y_n, \tag{5}$$

where y_n are so chosen as to satisfy condition (2).

Weigh now the analogue of this method for the solution of Eq. (3). Put

(6) $$a_n = \frac{f(z_n)\, f''(z_n)}{[f'(z_n)]^2} \qquad n = 0, 1, 2, \dots .$$

Then the sequence of approximate solutions z_n of Eq. (3) is defined by the following formula

(7) $$z_{n+1} = z_n - g(a_n)\frac{f(z_n)}{f'(z_n)}.$$

The general majorant principle

THEOREM. *Suppose that Eq.* (1) *has a real majorant equation* (3) *satisfying conditions* 1°—4°. *Let us assume that Eq.* (3) *has a positive root* z^* *lying to the right of* z_0. *Suppose now that the sequence of the approximate solutions* z_n *of Eq.* (3) *defined by* (7) *is increasing and it converges to* z^*. *Finally, suppose that the function* $g(z)$ *satisfies the following conditions*:

(8) $g(z)$ *is non-negative and non-decreasing for* $|z| \leqslant \frac{1}{2}$ *and*

(9) $$0 < a \leqslant g(t) \quad \textit{for} \quad |z| \leqslant \tfrac{1}{2}.$$

The function

(10) $$h(z) = 1 - g(z) + \tfrac{1}{2} z g^2(z)$$

is non-negative and non-decreasing for $0 \leqslant z \leqslant \frac{1}{2}$ *and*

(10′) $$|h(z)| \leqslant h(|z|) \quad \textit{for} \ |z| = \tfrac{1}{2}.$$

Then the sequence of approximate solutions x_n *defined by* (5) *converges to a solution* x^* *of Eq.* (1) *and we have the following error estimate*

(11) $$\|x^* - x_n\| \leqslant z^* - z_n.$$

Proof. Using the lemma of [5] we get

(12) $$a = \frac{f(z)\, f''(z)}{[f'(z)]^2} \leqslant \frac{1}{2} \qquad \text{for} \quad z_0 \leqslant z \leqslant z^*.$$

It follows from conditions 1°—3°, (4) and (6) that

(13) $$A_0 \leqslant a_0.$$

Hence, we get, by (12), 1°, 2°, (5) and (7),

(14) $$\|x_1 - x_0\| \leqslant z_1 - z_0,$$

since, by (8) and (13), we have

(15) $$g(A_0) \leqslant g(a_0).$$

We shall show that all conditions 1°—4° are satisfied if element x_0 is replaced by element x_1. Since $f''(z) \geqslant 0$, the derivative $f'(z)$ increases still preserving the minus sign at point z_1. We shall show that condition 2° is satisfied for $x = x_1$. We have

$$\|F'(x_1)\| \geqslant \|F'(x_0)\| \left(1 - \frac{\|F'(x_1) - F'(x_0)\|}{\|F'(x_0)\|}\right).$$

Using the abstract analogue of the fundamental formula of the integral calculus, we obtain, by 3° and (14),

$$\|F'(x_1)\| \geqslant F'(x_0)\| \left(1 - \frac{\left\|\int\limits_{x_0}^{x_1} F''(\bar{x})\, d\bar{x}\right\|}{\|F'(x_0)\|}\right) \geqslant \|F'(x_0)\| \left(1 - \frac{\int\limits_{z_0}^{z_1} f''(z)\, dz}{\|F'(x_0)\|}\right) \geqslant$$

$$\geqslant \|F'(x_0)\| \left(1 + \frac{f'(z_1) - f'(z_0)}{(z_0)}\right) = \|F'(x_0)\| \frac{f'(z_1)}{f'(z_0)}.$$

Hence, we get

$$\frac{1}{\|F'(x_1)\|} \leqslant -\frac{1}{f'(z_1)}.$$

It is easy to see that conditions 3° and 4° are also satisfied for x_1. In fact, if

$$\|x - x_1\| \leqslant z - z_1 \leqslant z' - z_1,$$

then we have, by (14),

$$\|x_1 - x_0\| \leqslant \|x - x_1\| + \|x_1 - x_0\| \leqslant z - z_1 + z_1 - z_0 = z - z_0 \leqslant z' - z_0.$$

We shall now show that condition 1° is satisfied for x_1. Using Taylor's formula in the integral form, we get, by (7),

$$f(z_1) = f(z_0) - g(a_0) f(z_0) + \frac{1}{2} f''(z_0) \frac{g^2(a_0) f^2(z_0)}{[f'(z_0)]^2} + \frac{1}{6} \int\limits_{z_0}^{z_1} f'''(z)\,(z_1 - z)^3\, dz\,.$$

Hence, we obtain, by (10),

$$f(z_1) = h(a_0) f(z_0) + \frac{1}{6} \int\limits_{z_0}^{z_1} f'''(z)\,(z_1 - z)^3\, dz. \tag{16}$$

On the other hand, using the analogue of the same formula, we get a similar expression for $F(x_1)$:

$$F(x_1) = h(A_0) F(x_0) + \frac{1}{6} \int\limits_{x_0}^{x_1} F'''(\bar{x})\,(x_1 - \bar{x})^3\, d\bar{x}, \tag{17}$$

where A_0 is as defined by (4). It follows from (10′), (12) and (13) that

$$|h(A_0)| \leqslant h(a_0). \tag{18}$$

Hence, we get, by (16), (17), (18) and 4°, that

$$|F(x_1)| \leqslant f(z_1).$$

In the same way as above we get the relation

$$\|x_2 - x_1\| \leqslant z_2 - z_1.$$

Thus, we obtain by induction

$$\|x_{n+p} - x_n\| \leqslant z_{n+p} - z_n \tag{19}$$

for any positive integers n and p.

Since the sequence of z_n converges to z^*, we conclude that there exists an element x^* of the sphere $S(x_0, r)$, $r = z' - z_0$, such that

$$x_n \to x^* \quad \text{as} \quad n \to \infty.$$

It follows from (19) that relation (11) is satisfied. It remains to prove that x^* is a solution of Eq. (1).

In virtue of (9) we have

$$\frac{F(x_n)}{F'(x_n) y_n} \to 0 \quad \text{as} \quad n \to \infty.$$

Since the sequence of the norms of $F'(x_n)$ is bounded, we have

$$F(x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, it follows from the continuity of $F(x)$ that x^* is a solution of Eq. (1).

Remark. There are Banach spaces which do not have the property that the norm of an arbitrary functional is reached at a point of the unit sphere. In such spaces condition (2) cannot be satisfied in general. In this case we can choose the elements y_n so as to satisfy the following condition instead of (2).

$$\|y_n\| \leqslant 1 \quad \text{and} \quad |f'(z_n)| \leqslant |F'(x_n) y_n| \leqslant \|F'(x_n)\|.$$

Then condition 2° should be replaced by the following one (see [6]).

$$\frac{1}{\|F'(x_0)\|} < B_0, \quad B_0 = -\frac{1}{f'(z_0)} > 0.$$

Application to Newton's metod

This case is trivial, since we have here

$$g(z) = 1 \quad \text{for any } z$$

and

$$h(z) = \tfrac{1}{2} z.$$

Application to Tchebyshev's method

In this case we have (see [4])

$$g(z) = 1 + \tfrac{1}{2} z$$

and

$$h(z) = \tfrac{1}{2} z^2 + \tfrac{1}{8} z^3.$$

Thus, we see that conditions (8)—(10′) of the Theorem are satisfied.

Application to the method of tangent hyperbolas

We have for this method (see [3])

$$g(z) = \frac{1}{1 - \frac{1}{2} z}$$

and the function (10) is of the form

$$h(z) = \frac{z^2}{4\left(1 - \frac{1}{2} z\right)^2}.$$

It is obvious that conditions (8)—(10′) of the Theorem are also satisfied in this case.

Application to Laguerre's method

In this case we have (see [7])

$$g(z) = \frac{N}{1 + (N-1)\sqrt{1 - \frac{N}{N-1} z}},$$

where N is a positive integer, $N \geqslant 3$.

Using Taylor's expansion for

$$\sqrt{1 - \frac{N}{N-1} z}$$

we get

$$h(z) = (N-2)\left[\frac{1}{2\cdot 4}\,\frac{N^2}{(N-1)}\,z^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}\,\frac{N^3}{(N-1)^2}\,z^3 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\,\frac{N^4}{(N-1)^3}\,z^4 + \dots\right]\cdot$$
$$\cdot\left[1 + (N-1)\sqrt{1 - \frac{N}{N-1} z}\right]^{-2}.$$

Hence, we see that all conditions (8)—(10′) of the Theorem are fulfilled.

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MATHEMATICS

An Iterative Method for the Eigenproblem of Linear Operators

by

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Presented by W. ORLICZ on July 20, 1391

1. This paper contains an iterative method for finding eigenvalues and eigenvectors of linear bounded and selfadjoint operators in Hilbert space. The method presented here is actually an extension of the method in [3] for computing eigenvalues and eigenvectors of real symmetric matrices. The general idea of the method is similar and is based on the generalization of Newton's method for finding zero-elements of a non-linear functional. The general theory of this generalization is given in [1] and [2]. As in [3], the examination of the method in question requires an independent argument. The proof of the convergence is slightly simplified here and does not use the spectral decomposition as in [3]. Besides, a generalization of the method is also given by introducing a parameter.

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. We shall assume operator A to be bounded and selfadjoint, i.e. $A^* = A$, where A^* is the adjoint of A.

The problem is to find eigenvalues λ and corresponding eigenvectors x of H satisfying the equation

$$Ax = \lambda x, \quad x \neq 0. \tag{1}$$

It follows from (1) that

$$\lambda(x) = \frac{(Ax, x)}{(x, x)}. \tag{2}$$

Hence, we get instead of Eq. (1) the following equation

$$Ax = \lambda(x)\, x, \quad x \neq 0.$$

Instead of this equation we shall solve the following functional equation

$$\|Ax - \lambda(x)\, x\|^2 = 0$$

or, equivalently,

$$F(x) = \|x\|^2 \, \|Ax\|^2 - (Ax, x)^2 = 0, \quad x \neq 0. \tag{3}$$

Thus, we see that our eigenproblem is equivalent to that of finding the solutions of Eq. (3). But this equation means that the value of cosine of the angle between the eigenvector x and its image Ax is equal to 1 or -1. Thus, we shall say that Eq. (3) is the cosine equation for the operator A. Hence, the eigenproblem of the operator A is equivalent to that of finding the solutions of the cosine equation for the operator A. But Eq. (3) is a functional equation with a non-linear functional being continuously differentiable in the sense of Fréchet. A method of solving such functional equations is given in [1] and [2].

We shall now apply this method to our eigenproblem. Denote by $F'(x)$ the Fréchet derivative of the functional $F(x)$ defined by (3). Then the sequence of approximate solutions x_n of Eq. (3) is defined as follows:

$$x_{n+1} = x_n - \frac{F(x_n)}{\|F'(x_n)\|^2} F'(x_n), \qquad n = 0, 1, 2, \ldots, \tag{4}$$

where x_0 is the initial approximate solution of Eq. (3).

Let us calculate the Fréchet derivative $F'(x)$ of $F(x)$ in (3).

We have, by (3),

$$F(x) = \|x\|^2 \left\| Ax - \frac{(Ax, x)}{\|x\|^2} x \right\|^2. \tag{5}$$

Using the abstract analogues of the differential chain rules, we obtain

$$y = \tfrac{1}{2} F'(x) = \|x\|^2 \left(A - \frac{(Ax, x)}{\|x\|^2} I\right)^2 x + \left\| \left(A - \frac{(Ax, x)}{\|x\|^2} I\right) x \right\|^2 x. \tag{6}$$

Hence, we get

$$y = \tfrac{1}{2} F'(x) = (\|x\|^2 B_x^2 + \|B_x x\|^2 I)\, x, \tag{7}$$

where

$$B_x = A - \frac{(Ax, x)}{\|x\|^2} I$$

and I denotes the identity mapping. It is easy to see that the expression (7) is equivalent to the following one:

$$y = \tfrac{1}{2} F'(x) = \|Ax\|^2 x + \|x\|^2 A^2 x - 2\,(Ax, x)\, Ax. \tag{8}$$

Thus, we get instead of (4)

$$x_{n+1} = x_n - \frac{\|x_n\|^2 \|Ax_n\|^2 - (Ax_n, x_n)^2}{2\,\|y_n\|^2} y_n, \tag{9}$$

where y_n is defined by formula (6) or (8) wherein x should be replaced by x_n.

We shall now establish some relations needed in the sequel. In virtue of (5) and (6) we have

$$(y, x) = 2F(x). \tag{10}$$

Since

$$B_x A = AB_x,$$

relations (5) and (6) imply

$$|(Ay, x)| \leqslant 2\,\|A\|\,F(x). \tag{11}$$

We shall now show the convergence of the sequence of $\|x_n\|^2$. It follows from (9) and (10) that

$$\|x_{n+1}\|^2 = \|x_n\|^2 - \frac{7}{4}\,\frac{F^2(x_n)}{\|y_n\|^2}. \tag{12}$$

Hence, it follows that the sequence of $\|x_n\|^2$ is decreasing and bounded and, consequently, convergent. Relation (12) implies

$$\|x_{n+1}\|^2 = \|x_0\|^2 - \frac{7}{4}\sum_{i=0}^{n}\frac{F^2(x_i)}{\|y_i\|^2}, \tag{13}$$

Whence follows the convergence of the series

$$\sum_{i=0}^{\infty}\frac{F^2(x_i)}{\|y_i\|^2} < \infty. \tag{14}$$

From (13) we have that

$$x_n \to 0 \quad \text{as} \quad n \to \infty$$

if and only if

$$\|x_0\|^2 = \frac{7}{4}\sum_{i=0}^{\infty}\frac{F^2(x_i)}{\|y_i\|^2}. \tag{15}$$

We shall now prove the convergence of the sequence of numbers (Ax_n, x_n). We have, by (9),

$$(Ax_{n+1}, x_{n+1}) = \left(Ax_n - \frac{F(x_n)}{2\,\|y_n\|^2}\,Ay_n,\; x_n - \frac{F(x_n)}{2\,\|y_n\|^2}\,y_n\right).$$

Hence, we get

$$(Ax_{n+1}, x_{n+1}) - (Ax_n, x_n) = \frac{F^2(x_n)}{4\,\|y_n\|^2}\left[\frac{(Ay_n, y_n)}{\|y_n\|^2} - 4\,\frac{(Ay_n, x_n)}{F(x_n)}\right]. \tag{16}$$

We shall show that the expression in square brackets in (16) is bounded. In fact, we have

$$\frac{|(Ay_n, y_n)|}{\|y_n\|^2} = \|A\| \qquad \text{for} \qquad n = 0, 1, 2, \ldots. \tag{17}$$

It follows from (11) that

$$\frac{|(Ay_n, x_n)|}{F(x_n)} \leqslant 2\,\|A\|. \tag{18}$$

If $\|y_n\|$ in (17) or $F(x_n)$ in (18) vanishes then, of course, x_n is an eigenvector of A. Thus, it follows from (17) and (18) that the expression in square brackets in (16)

is bounded. In virtue of (16), the convergence of the series in (14) implies the same for the following series:

$$\sum_{n=0}^{\infty} (Ax_{n+1}, x_{n+1}) - (Ax_n, x_n) = \lim_{k\to\infty} (Ax_k, x_k) - (Ax_0, x_0).$$

Thus, the convergence of the sequence of numbers (Ax_n, x_n) has been proved.

Suppose now that element x_0 is so chosen that condition (15) is not satisfied. Then the sequence of $\|x_n\|^2$ converges and its limit is different from zero. Hence, it follows that the sequence

$$\left\{\frac{(Ax_n, x_n)}{\|x_n\|^2}\right\}, \qquad n = 0, 1, 2, \dots$$

is convergent. Denote its limit by λ. The question is now whether λ is an eigenvalue of A. Since the sequence of $\|y_n\|^2$ is bounded it follows that

$$F(x_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{19}$$

Since the sequence of $\|x_n\|$ is bounded, there exists a subsequence of $\{x_n\}$ weakly convergent to some element x^*. If x^* is different from the zero-element, then it follows from (19) and (5) that

$$\|Ax^* - \lambda x^*\| = 0. \tag{20}$$

Hence, x^* is an eigenvector of A. It is easy to see that if λ is a simple eigenvalue of A then the sequence of x_n converges weakly towards the eigenvector x^* corresponding to λ, provided that (x_n) does not converge weakly to 0.

Suppose now, in addition, that operator A is completely continuous. Then there exists a subsequence of $\{x_n\}$ strongly convergent to an eigenvector x^* corresponding to λ. If in this case λ is a simple eigenvalue of A, then the sequence of x_n converges strongly to the eigenvector x^* corresponding to λ. Thus, we have proved the following

THEOREM. *Let x_0 be an arbitrary element such that condition* (15) *is not satisfied and let x_n be defined by the process in* (9). *Then the sequence of numbers* $\dfrac{(Ax_n, x_n)}{\|x_n\|^2}$ *converges to a number λ and the following condition is satisfied*

$$Ax_n - \frac{(Ax_n, x_n)}{\|x_n\|^2} x_n \to 0 \quad \textit{as} \quad n \to \infty.$$

If in addition operator A is completely continuous, then λ is an eigenvalue of A and the sequence of x_n converges strongly to an eigenvector x^* corresponding to λ.

2. We shall now introduce a parameter α in the process defined by (4). Thus, we get instead of (9)

$$x_{n+1} = x_n - \alpha \frac{\|x_n\|^2 \|Ax_n\|^2 - (Ax_n, x_n)^2}{2\|y_n\|^2} y_n, \tag{21}$$

where y_n is defined by formula (6) or (8) wherein x should be replaced by x_n Then we get the following relation instead of (12):

$$\|x_{n+1}\|^2 = \|x_n\|^2 - 2\alpha\left(1 - \frac{1}{8}\alpha\right)\frac{F^2(x_n)}{\|y_n\|^2}.$$

Hence, it follows that for $0 < \alpha < 8$ the sequence of $\|x_n\|^2$ is convergent and so is the series in (14). We can now apply the same argument as in par. 1. But we should replace the number 7/4 in relation (15) by $2\alpha\,(1 - 1/8\,\alpha)$. Then the Theorem of par. 1 remains also true for the process defined by (21). Let us observe that $\|x_{n+1}\|^2$ is minimized for $\alpha = 4$.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
Серия мат., астр. и физ. наук
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МАТЕМАТИКА

О. М. ФОМЕНКО

ОЦЕНКИ НЕКОТОРЫХ ТРИГОНОМЕТРИЧЕСКИХ СУММ

Представлено В. СЕРПИНСКИМ 10 августа 1961

Обозначения. N — достаточно большое положительное число; $r = \ln N$; ε — как угодно малое положительное фиксированное число; l, m, q, y — целые положительные числа; a — целое число; $u_1 < u_2 < \ldots < u_{m-1}$ — последовательность, состоящая из $m-1$ фиксированных четных положительных чисел; $p, p^{(1)}, p^{(2)}, \ldots$ — простые числа; $\Omega(y)$ — число простых делителей числа y.

В настоящей заметке оцениваются простейшие тригонометрические суммы, когда аргумент пробегает такие простые числа p, что $p+u_1, p+u_2, \ldots, p+u_{m-1}$ — тоже простые. Оценки подобного типа для случая обычных простых чисел даны И. М. Виноградовым [1]. Они играют важную роль при решении проблемы Гольдбаха. Для случая „близнецов" аналогичные оценки даны Юе Минь-и [2]. Ряд его приемов использован в настоящей работе.

Теорема 1. *Пусть* $N e^{-r^{1-2\varepsilon}} \leqslant A \leqslant \frac{1}{2} N$; $(a, q) = 1$; $0 < q \leqslant e^{r^{\varepsilon}}$; *положим*

$$S = \sum_{\substack{N-A < p \leqslant N \\ p+u_1 = p^{(1)} \\ \cdots\cdots \\ p+u_{m-1} = p^{(m-1)}}} e^{2\pi i \frac{a}{q} p};$$

тогда

$$S \ll \frac{A (rq)^{(2m+3)\varepsilon}}{r^m \sqrt{q}}, \tag{1}$$

где входящая в символ $\ll$ *постоянная зависит от величин* $u_1, u_2, \ldots, u_{m-1}, m, \varepsilon$.

Наметим доказательство теоремы. Оно состоит в комбинировании метода тригонометрических сумм с решетом Сельберга (см. [3], [4]).

Пусть $b_0 = e^{r^{1-1,5\varepsilon}}$ и P — произведение всех простых чисел, не превосходящих b_0 и не делящих q. Имеем

$$S = \sum_{\substack{N-A < y \leqslant N \\ (y(y+u_1)\ldots(y+u_{m-1}), Pq)=1}} e^{2\pi i \frac{a}{q} y} - \sum_{k=1}^{m-1} \sum_{\substack{N-A < p \leqslant N \\ \Omega(p+u_k) > 1 \\ ((p+u_1)\ldots(p+u_{m-1}), Pq)=1}} e^{2\pi i \frac{a}{q} p} - \sum_{\substack{N-A < y \leqslant N \\ \Omega(y) > 1 \\ (y(y+u_1)\ldots(y+u_{m-1}), Pq)=1}} e^{2\pi i \frac{a}{q} y} =$$

$$= S_1 - \sum_{k=1}^{m-1} S'_k - S'',$$

причем для S'_k числа $p+u_1, \ldots, p+u_{k-1}$ — простые. $S'_k\,(k=1, 2, \ldots, m-1)$ и S'' оцениваются аналогично. Условие $\Omega(p+u_k)>1$ (и $\Omega(y)>1$ для S'') позволяет легко свести оценку этих сумм к оценке двойных сумм, причем важное значение имеет лемма 9 работы [2] и применение метода решета. Особенно удобно использовать новые результаты Н. И. Климова по решету Сельберга (см. [3], Теорема 3).

S_1 приходится оценивать иначе. Легко видеть, что

$$S_1 = \sum_{\substack{1\leqslant l<q \\ (l(l+u_1)\ldots(l+u_{m-1}),\, q)=1}} e^{2\pi i \frac{a}{q} l} \sum_{\substack{N-A<y\leqslant N \\ (y(y+u_1)\ldots(y+u_{m-1}),\, P)=1 \\ y\equiv l \pmod q}} 1 .$$

Для дальнейшего вводим и оцениваем сумму

$$S(a, q) = \sum_{\substack{1\leqslant l<q \\ (l(l+u_1)\ldots(l+u_{m-1}),\, q)=1}} e^{2\pi i \frac{a}{q} l} .$$

Довольно легко преобразовать $S(a, q)$ в произведение, а затем показать, что

$$S(a, q) \leqslant m^{\Omega(q)} \ll q^{\varepsilon} .$$

Обобщая рассуждение лемм 3 и 4 работы [2], можно показать, что

$$\sum_{\substack{N-A<y\leqslant N \\ (y(y+u_1)\ldots(y+u_{m-1}),\, P)=1 \\ y\equiv l \pmod q}} 1 = \sum_{\substack{N-A<y\leqslant N \\ (y(y+u_1)\ldots(y+u_{m-1})\, P)=1 \\ y\equiv 1 \pmod q}} 1 + O\left(\frac{A}{q^2}\, e^{-r^{\varepsilon/2}}\right).$$

Отсюда

$$S_1 = \sum_{\substack{1\leqslant l<q \\ (l(l+u_1)\ldots(l+u_{m-1}),\, q)=1}} e^{2\pi i \frac{a}{q} l} \sum_{\substack{N-A<y\leqslant N \\ (y(y+u_1)\ldots(y+u_{m-1}),\, P)=1 \\ y\equiv 1 \pmod q}} 1 + O\left(\frac{A}{q}\, e^{-r^{\varepsilon/2}}\right). \tag{2}$$

Дла оценки S_1 остается оценить вторую суммы в (2). Она оценивается методом решета Сельберга с помощью приемов, вполне аналогичных известным.

Из Теоремы 1 следует

Теорема 2. *Пусть* $1<H\leqslant e^{r^{\varepsilon}}$; $\tau = NH^{-1}$; $\alpha = \dfrac{a}{q} + z$; $(a, q) = 1$; $0<q\leqslant e^{r^{0,5\varepsilon}}$; $|z| \leqslant \dfrac{1}{q\tau}$; *положим*

$$S = \sum_{\substack{p\leqslant N \\ p+u_1=p^{(1)} \\ \cdots\cdots \\ p+u_{m-1}=p^{(m-1)}}} e^{2\pi i \alpha p} .$$

Тогда

$$S \ll \frac{N(rq)^{(2m+3)\varepsilon}}{r^m \sqrt{q}}, \tag{3}$$

где входящая в символ $\ll$ постоянная зависит от величин $u_1, u_2, \ldots, u_{m-1}, m, \varepsilon$.

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O. M. FOMENKO, ESTIMATIONS OF SOME TRIGONOMETRICAL SUMS

The object of this note is to prove the following

THEOREM. *Let* $\varepsilon > 0$ *be any given number*, $N > 2$; $r = \ln N$; $1 < H \leqslant e^{r^\varepsilon}$; $\tau = NH^{-1}$; $\alpha = \frac{a}{q} + z$; $(a, q) = 1$; $0 < q \leqslant e^{r^{0,5\varepsilon}}$, $|z| \leqslant \frac{1}{q\tau}$; $0 < u_1 < u_2 < \ldots < u_{m-1}$ — *even integers;*

$$S = \sum_{\substack{p \leqslant N \\ p+u_1 = p^{(1)} \\ \cdots\cdots \\ p+u_{m-1} = p^{(m-1)}}} e^{2\pi i \alpha p},$$

where $p, p^{(1)}, \ldots, p^{(m-1)}$ *denote primes, then*

$$S \ll \frac{N(rq)^{(2m+3)\varepsilon}}{r^m \sqrt{q}}$$

(*the constant implied by the symbol* $\ll$ *is dependent on* $u_1, u_2, \ldots, u_{m-1}, m, \varepsilon$).

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MATHEMATICS

The Existential and Universal Statements on Parallels

by

Z. PIESYK

Presented by K. BORSUK on August 14, 1961

Let $\mathscr{A}$ be the plane geometry based upon the first three groups of Hilbert's axioms (see [4]), i.e. upon the plane axioms of incidence, axioms of order, and axioms of congruence. We say that a *point a is parabolic with respect to a line L*, in symbols — $P(a, L)$, if there is exactly one line K passing through a and not intersecting L, and that *a is hyperbolic with respect to L*, in symbols — $H(a, L)$, if there are at least two distinct lines K_1 and K_2 both passing through a and not intersecting L. We consider the sentences:

Φ_P: $$\bigvee_{a,L} P(a, L) \rightarrow \bigwedge_{a \notin L} P(a, L),$$

Φ_H: $$\bigvee_{a,L} H(a, L) \rightarrow \bigwedge_{a \notin L} H(a, L).$$

(The restricted general quantifier $\bigwedge\limits_{a \notin L}$ is read *for every point a and for every line L such that $a \notin L$*. In general, for the meaning of the logical symbols see p. 762). R. Baldus [2] proved the implication Φ_H in the theory $\mathscr{A}_1$ which arises from $\mathscr{A}$ by supplementing the axiom system with the Archimedean postulate, thus in a non-elementary theory. P. Szasz [6] proved Φ_H in the elementary theory $\mathscr{A}_2$ which arises from $\mathscr{A}$ by supplementing the axiom system with the axiom on the existence of the parallel. W. Schwabhauser [5] proved Φ_H in the theory $\mathscr{A}_3$ which differs from $\mathscr{A}_2$ only in that the axiom on the existence of the parallel is replaced by another elementary instance of the continuity axiom. In this paper we shall show that Φ_H is a theorem of geometry $\mathscr{A}$ itself*). Obviously, Φ_H implies at once Φ_P in $\mathscr{A}$.

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*) This result (due to the author) was presented by Dr W. Szmielew at the Colloquium on Foundations of Geometry in Oberwolfach, April 13—18, 1961. It was communicated during the Colloquium that another proof of Φ_H in $\mathscr{A}$ due to Dr J. Strommer is in press in Hungary. Moreover, during Colloquium, Dr R. Lingenberg succeded in proving Φ_H in the Bachmann system of geometry (see [1]) which is weaker that $\mathscr{A}$. Since the proofs of Doctors Strommer and Lingenberg are based on Hjelmslev's ideas, and the one presented in this paper involves only the simplest geometrical notions like the triangle and the circle, the author hopes that his result remains of interest by itself.

By a *segment ab* we understand any non-ordered couple of distinct points a and b. By a *triangle abc* we understand any non-ordered triple of non-collinear points a, b, c. We denote by $\boldsymbol{L}(ab)$ the line ab passing through the points a and b $(a \neq b)$, and by (ab) the open segment ab, i.e. the set of all points p which lie between the points a and b $(a \neq b)$. We denote by AB the angle, the sides of which coincide with the half-lines A and B, and by $\boldsymbol{L}(A)$ — the line including the half-line A. The symbol $\equiv$ denotes the congruence relation. We shall say that a *segment ab is parabolic*, in symbols — $\boldsymbol{P}(ab)$, if the point a is parabolic with respect to the line L passing through b and perpendicular to the line ab. We shall say that a *triangle abc is parabolic*, in symbols — $\boldsymbol{P}(abc)$, if the sum of its three (inner) angles equals two right angles. We use the following logical constants: $\rightarrow$ for implication, $\sim$ for negation, $\wedge$ for conjunction, $\vee$ for disjunction, $\leftrightarrow$ for equivalence, $\bigwedge$ and $\bigvee$ for universal and existential quantifiers. Moreover, we use the symbol $\in$ for the membership relation, the symbol $\cap$ for the set-theoretical product, and 0 for the empty set.

The proof is based upon eight simple lemmas provable in $\mathscr{A}$.

LEMMA 1. $\boldsymbol{P}(a, L) \rightarrow \bigwedge\limits_{b,c\in L} (b \neq c \rightarrow \boldsymbol{P}(abc))$.

The proof is obvious.

LEMMA 2. $\bigvee\limits_{abc} \boldsymbol{P}(abc) \rightarrow \bigwedge\limits_{abc} \boldsymbol{P}(abc)$.

For proof see, e.g. [3].

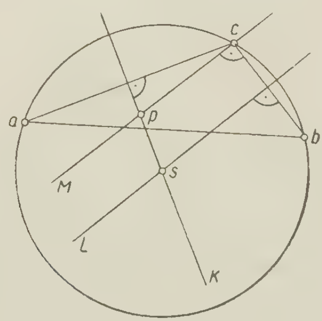

Fig. 1

LEMMA 3. $\boldsymbol{P}(ab) \wedge (abc \equiv cd \vee ab > cd) \rightarrow \boldsymbol{P}(cd))$.

The proof is obvious.

LEMMA 4. $\boldsymbol{P}(a, L) \wedge \boldsymbol{L}(ab) \cap L = 0 \rightarrow \boldsymbol{P}(b, L)$.

It is an immediate consequence of the definition of $\boldsymbol{P}$ and Lemmas 1-3.

LEMMA 5. $\bigwedge\limits_{abc} (\boldsymbol{P}(ac) \wedge \boldsymbol{P}(bc) \rightarrow \bigvee\limits_{s} (sa \equiv sb \equiv sc))$. (*If two sides of a triangle abc are parabolic, then there is a circle S with the centre s such that $a, b, c \in S$*).

Proof. Consider a triangle abc such that $\boldsymbol{P}(ac)$ and $\boldsymbol{P}(bc)$ (Fig. 1). Let K and L be the perpendicular bisectors of the sides ac and bc, respectively. Let M be the line

through the vertex c perpendicular to the line bc. Then $\boldsymbol{P}(c, K)$, by Lemma 3, and, consequently, M intersects K in a point p. Furthermore, $P(p, L)$, by Lemmas 3 and 4, and, consequently, K intersects L in a point s. Obviously, $sa \equiv sb \equiv sc$.

LEMMA 6. $\bigwedge\limits_{pqr} \boldsymbol{P}(pqr) \rightarrow \bigwedge\limits_{abc} \bigwedge\limits_{s,d} (sa \equiv sb \equiv sc \wedge \boldsymbol{L}(ab) \cap (cd) = 0 \rightarrow$ $(\sphericalangle acb \equiv \sphericalangle adb \leftrightarrow sc \equiv sd))$.

(By the assumption that every triangle is parabolic, given a triangle abc, a circle S with the centre s such that a, b, c ∈ S, and a point d, if points c and d lie on the same side of the line ab, then d ∈ S if and only if $\sphericalangle acb \equiv \sphericalangle adb$*).*

The proof is obvious.

To formulate the last two lemmas let us consider the sentence

Ψ: *For any arbitrary angle AB, for any half-line C with an origin c, and for any point* $d \notin \boldsymbol{L}(C)$, *if* $\boldsymbol{L}(cd) \perp \boldsymbol{L}(C)$, *then there is a point x on C such that* $\sphericalangle cxd < AB$.

We shall prove in $\mathscr{A}$ the following two lemmas:

LEMMA 7. $\bigvee\limits_{a,L} \boldsymbol{P}(a, L) \wedge \bigvee\limits_{d,M} \boldsymbol{H}(d, M) \rightarrow \sim \Psi$.

LEMMA 8. $\bigvee\limits_{a,L} \boldsymbol{P}(a, L) \rightarrow \Psi$.

Proof of Lemma 7. We assume that $\bigvee\limits_{a,L} \boldsymbol{P}(a, L)$. In consequence, by Lemmas 1 and 2, every triangle is parabolic. Moreover, we assume that $\boldsymbol{H}(d, M)$ for a point d and for a line M (Fig. 2). Let us denote by c the perpendicular projection of d upon M, by K the line through d perpendicular to the line dc, and by W one of the half-plane with the boundary dc; furthermore let $B = W \cap K$, $C = W \cap M$. Since $\boldsymbol{H}(d, M)$, there is a half-line A with the origin d between the half-lines B and dc such that $A \cap C = 0$. Obviously, $\sphericalangle cxd > AB$ for every point x on C. Thus $\sim \Psi$.

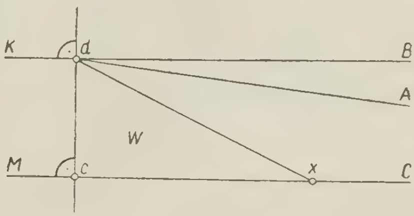

Fig. 2

Proof of Lemma 8. We assume that $\bigvee\limits_{a,L} \boldsymbol{P}(a, L)$. Hence, there are parabolic segments and, by Lemmas 1 and 2, every triangle is parabolic. We consider now an arbitrary acute angle AB (it is sufficient to prove Ψ in case AB is acute), as well as an arbitrary half-line C, with the origin c and a point $d \notin \boldsymbol{L}(C)$ such that $\boldsymbol{L}(cd) \perp \boldsymbol{L}(C)$ (Fig. 3). We look for a point x on C such that $\sphericalangle cxd < AB$. Let W be the half-plane with the boundary cd including C, let A_1 be the half-line complementary to the half-line cd, and let B_1 be the half-line in W with the origin c and such that $A_1 B_1 \equiv AB$. We pick a point p on B_1 to obtain a parabolic segment cp

(see Lemma 3). The half-line C intersects the open segment dp in a point q. We take points $a \in (pc)$ and $b \in (pq)$ in such a way that $\sphericalangle\, acb \equiv \sphericalangle\, adb$. In the triangle abc the sides ac and ab are, by Lemma 3, parabolic. Consequently, by Lemma 5, the vertices a, b, c lie on a circle S, and by Lemma 6 the point d is also on S. It is easily seen that the half-line C intersects S in a point x. Using again Lemma 6 we get $\sphericalangle\, cxd \equiv \sphericalangle\, cad$, and since $\sphericalangle\, cad < A_1 B_1$ and $A_1 B_1 \equiv AB$ we conclude that $\sphericalangle\, cxd < AB$. Thus Ψ.

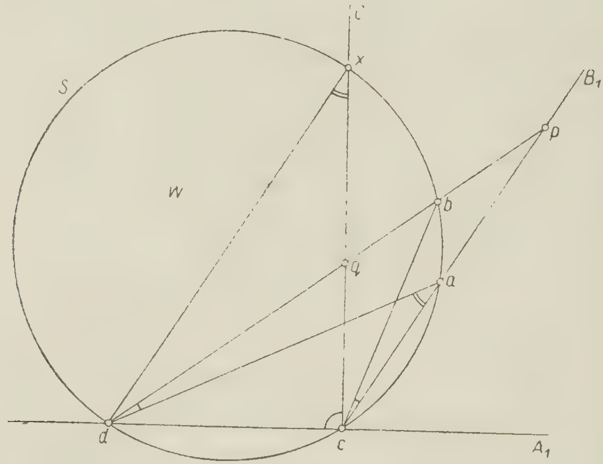

Fig. 3

From Lemmas 7 and 8 we derive at once Φ_P and Φ_H by a purely logical argument. In conclusion we get

STATEMENT. *The sentences Φ_P and Φ_H are theorems of geometry $\mathcal{A}$.*

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БЮЛЛЕТЕНЬ ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

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Д. МАНЖЕРОН и Л. Е. КРИВОШЕИН, **О НЕКОТОРЫХ ПРИБЛИЖЕННЫХ ЗАДАЧАХ, ОТНОСЯЩИХСЯ К ОДНОМУ НОВОМУ КЛАССУ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ** . . . . стр. 707—712

В настоящей заметке, приводя некоторые сжатые доказательства, авторы устанавливают приближенные оценки допускаемых погрешностей при неточных решениях граничных задач (1), (2) или же (3), (2), сводящихся, соответственно, к эквивалентным интегральным уравнениям (4), (5) или же (19), (M), причем функция N, входящая в это последнее, определяется, следуя ходу мыслей работы [10], выражениями (M). Искомые приближенные оценки даны в (10) или же в (17).

Статья, тесно примыкающая к ряду граничных задач для интегро-дифференциальных уравнений в „полных производных" в смысле М. Пиконе [1], составляющих содержание статьи, находящейся в печати в Известиях Ясского Политехнического Института, заканчивается некоторыми указаниями о возможных обобщениях как по отношению к числу независимых переменных, так и по отношению к виду граничных условий, самих интегро-дифференциальных уравнений или же форм и свойств функций (M), обобщающих функции Грина для рассматриваемых новых классов неэллиптических уравнений.

В. ПОГОЖЕЛЬСКИЙ, **СВОЙСТВА ИТЕРИРОВАННЫХ СИНГУЛЯРНЫХ ИНТЕГРАЛОВ В ПРОСТРАНСТВЕ** . . . . стр. 713—719

В работе автором доказаны три теоремы, касающиеся итерированных сингулярных интегралов в пространстве. В первой и второй теореме автор доказывает формулы перестановки (4), (26) интегралов, в которых один содержит функцию $G(z-y, z)$ с сильной особенностью для $z=y$, а другой — функцию $\Psi(x, z)$, определенную формулой (51) со слабой особенностью для $x=z$; функция $f(x)$ принадлежит к классу $\mathfrak{H}_x^h$. Третья теорема кассается формулы перестановки (27) интегралов с двума сильными особенностями $d=z$ и $z=x$; функция $\Phi(x)$ определена формулой (31).

В работе дается внутреннее определение вполне регулярных аналитических пространств, а также приводится необходимое и достаточное условие для того, чтобы вполне регулярное пространство было образом пространства всех иррациональных чисел при некотором непрерывном отображении.

В работе предложен итерационный метод нахождения собственных чисел и собственных векторов действительных симметричных матриц. Метод опирается на обобщение метода Ньютона, данное автором для нахождения нулевых элементов нелинейных функционалов в пространстве Банаха.

Работа содержит итерационный метод решения линейного операторного уравнения в пространстве Гильберта. Сущность этого метода состоит в введении вспомогательного подпространства. Таким образом, итерационный процесс происходит в некотором подпространстве. При удачном выборе вспомогательного подпространства скорость сходимости процесса может значительно улучшиться. Дается также применение к решению систем линейных алгебраических уравнений.

Работа содержит итерационный метод решения линейных уравнений в пространстве Гильберта. Для улучшения сходимости процесса вводится вспомогательное подпространство. При удачном выборе вспомогательного подпространства скорость сходимости метода может быть гораздо больше, чем в обычном методе с минимальными невязками. Дается также применение к решению систем линейных алгебраических уравнений.

В настоящей работе устанавливается один общий принцип мажорант для целого ряда итерационных методов высшего порядка. Как частные случаи применения этого принципа получаются методы: Ньютона, Чебышева, касательных гипербол и Лагерра. Таким образом, все перечисленные методы можно рассматривать с одной токчи зрения. По существу определяется класс методов, содержащий все перечисленные методы, точнее, их абстрактные варианты для функционалов.

М. АЛЬТМАН, **ИТЕРАЦИОННЫЙ МЕТОД ДЛЯ СОБСТВЕННОЙ ПРОБЛЕМЫ ЛИНЕЙНЫХ ОПЕРАТОРОВ** . . . . стр. 757—760

Работа содержит распространение на случай линейных ограниченных операторов в пространстве Гильберта одного метода автора нахождения приближенных собственных значений и собственных векторов матриц. Кроме того, рассматривается также обобщение этого метода, полученное путем введения численного параметра.

З. ПЕСЫК, **ЭКЗИСТЕНЦИАЛЬНЫЕ И УНИВЕРСАЛЬНЫЕ ПРЕДЛОЖЕНИЯ О ПАРАЛЛЕЛЬНЫХ** . . . . стр. 761—764

Пусть $\mathcal{A}$ обозначает плоскую абсолютную геометрию, опирающуюся на три первые группы аксиом Гильберта, т. е. на плоских аксиомах инциденции, аксиомах упорядочения и аксиомах конгруэнтности. Пусть $\boldsymbol{P}(a, L)$ обозначает, что через пункт a проходит лишь одна прямая L', не пересекающая L.

Предложение

$$\bigvee_{a,\,L} \boldsymbol{P}(a, L) \rightarrow \bigwedge_{a \notin L} \boldsymbol{P}(a, L)$$

является теоремой геометрии $\mathcal{A}$. Символы $\bigvee$ и $\bigwedge$ обозначают, соответственно, частный и общий кванторы, символ $\rightarrow$ обозначает импликацию.

TABLE DES MATIÈRES

Mathématique

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